Episode 34: Power series: radius and interval of convergence

$y = x^n$ power fn

p.s. in $(x-a)$
around a
centered at a
about a

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$$

c_n: given numbers (coefficients of p.s.)

x: a variable

a: given number (center of p.s.)

Ex. 1

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{1}{n!}(x-0)^n$$

$\frac{1}{n!}$: coeff. 0: center

p.s. centered at 0 with coeff. $c_n = \frac{1}{n!}$

Ex. 2

$$\sum_{n=1}^{\infty} \frac{(-3)^n (x+4)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-3)^n}{\sqrt{n}} (x - \underbrace{(-4)}_{\text{center}})^n$$

$\frac{(-3)^n}{\sqrt{n}}$: coeff. c_n

Ex. 3

$$\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} 1 \cdot (x-0)^n = 1 + x + x^2 + \dots$$

1: coeff. c_n 0: center

Does a p.s. conv. / div. ?

depends on a, c_n, x

Any p.s. conv. for $x = a$:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} c_n \underbrace{(a-a)}_{0}^n = \underline{0}$$

Any p.s conv. at its center.

Theorem (convergence of p.s.)

For a p.s. $\sum_{n=0}^{\infty} c_n (x-a)^n$

One of the following alternatives holds true:

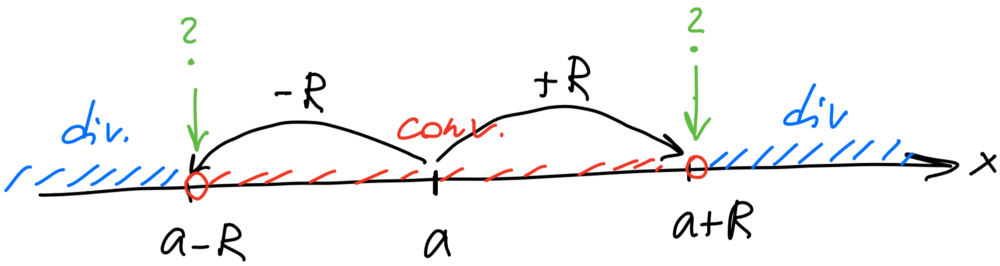
$R=0$ 1. p.s. conv. only for $x=a$

$R=\infty$ 2. p.s. conv. for all x

$0<R<\infty$ 3. There exists a number $R>0$ s.t.

- p.s. conv. for any x s.t. $|x-a|<R$
- p.s. div. for any x s.t. $|x-a|>R$

Def. The number R is called the radius of convergence of p.s.



$$|x-a|<R \iff -R<x-a<R \iff a-R<x<a+R$$

$(a-R, a+R)$ is called the interval of conv.

p.s. div. for all x $\quad |x-a|>R$

Endpoint $a-R$, $a+R$ are studied separately

Proof Apply the ratio test for p.s. $\displaystyle\sum_{n=0}^{\infty} \underbrace{c_n (x-a)^n}_{a_n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{c_{n+1}(x-a)^{n+1}}{c_n (x-a)^n}\right| = \underbrace{\left|\frac{c_{n+1}}{c_n}\right|}_{\substack{\downarrow n\to\infty \\ L}} \cdot |x-a| \xrightarrow[n\to\infty]{} L\cdot|x-a|$$

L (may be 0 or ∞)

By the ratio test,

p.s. conv. if $\quad L\cdot|x-a|<1 \iff |x-a|<\frac{1}{L}=R$

p.s. div. if $\quad L\cdot|x-a|>1 \iff |x-a|>\frac{1}{L}=R$

$L = 0 \Rightarrow R = \infty$
$L = \infty \Rightarrow R = 0$

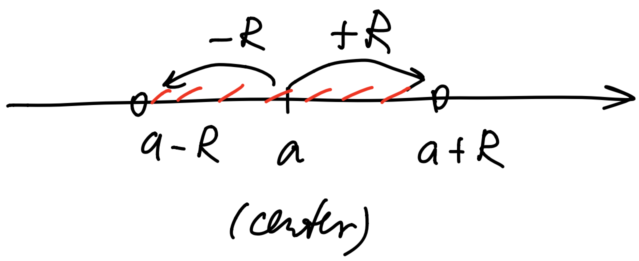
How to find the radius and the interval of conv.?

1. Bring p.s. to the standard form $\sum_{n=0}^{\infty} c_n (x-a)^n$

2. Find $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = L$ (may be 0 or ∞)

3. The radius of conv. is $R = \frac{1}{L}$ $\left(\begin{matrix} L=0 \Rightarrow R=\infty \\ L=\infty \Rightarrow R=0 \end{matrix}\right)$

4. The interval of conv. is $\underbrace{(a-R,\ a+R)}_{\text{at least}}$



5. Study endpts of the interval of conv.
$x = a-R$
$x = a+R$

(include / not include into the interval)

Ex. For p.s. $\sum_{n=1}^{\infty} \frac{(4x+2)^n}{n}$ find the center, the radius of conv., the interval of conv.

Sol.
1) Bring p.s. to st. form
$\frac{(4x+2)^n}{n}$ should look like $c_n(x-a)^n$

$$\frac{(4x+2)^n}{n} = \frac{(4(x+\frac{1}{2}))^n}{n} = \frac{4^n}{n}(x+\tfrac{1}{2})^n$$

$x-(-\frac{1}{2})$

the center is $x = -\frac{1}{2}$ center

the coeff. are $c_n = \dfrac{4^n}{n}$

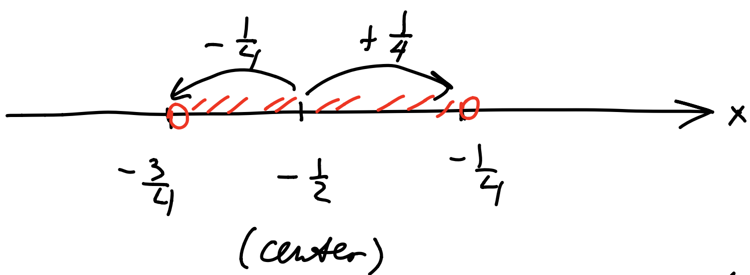
2,3) Find the radius

$$\left|\frac{c_{n+1}}{c_n}\right| = \frac{4^{n+1} \cdot n}{(n+1)\cdot 4^n} = \frac{4n}{n+1} = \frac{4}{1+\frac{1}{n}} \xrightarrow[n\to\infty]{} 4 = L$$

$\frac{1}{n} \to 0$

the radius is $R = \dfrac{1}{L} = \dfrac{1}{4}$

4) Find the interval of conv.



$-\frac{1}{2} + \frac{1}{4} = -\frac{1}{4}$

$-\frac{1}{2} - \frac{1}{4} = -\frac{3}{4}$

the interval of conv. is, at least, $\left(-\frac{3}{4}, -\frac{1}{4}\right)$

5) Study endpts $x = -\frac{3}{4}$, $x = -\frac{1}{4}$

Plug in $x = -\frac{3}{4}$ into original p.s :

$$\sum_{n=1}^{\infty} \frac{\left(4\cdot\left(-\frac{3}{4}\right)+2\right)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$ alt. harm. series converges

Plug in $x = -\frac{1}{4}$:

$$\sum_{n=1}^{\infty} \frac{\left(4\left(-\frac{1}{4}\right)+2\right)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$$ harm. series diverges

So the interval of conv. is $\left[-\frac{3}{4}, -\frac{1}{4}\right)$

Ex. 2 $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ int. of conv. ?

$$= \sum_{n=0}^{\infty} \underbrace{\left(\frac{1}{n!}\right)}_{C_n} (x-\underset{\uparrow \text{ center}}{0})^n.$$

$$\left|\frac{C_{n+1}}{C_n}\right| = \frac{1}{(n+1)!} \div \frac{1}{n!} = \frac{n!}{(n+1)!} = \frac{1}{n+1} \xrightarrow[n\to\infty]{} 0 = L$$

The radius is $R = \frac{1}{L} = \infty$

the interval of conv. is $(-\infty, \infty)$

p.s. conv. for all $\underline{\underline{x}}$

Ex. 3 $\sum_{n=0}^{\infty} n!\, x^n$ int. of conv. ?

center : $x=0$

coeff. : $C_n = n!$

$$\left|\frac{C_{n+1}}{C_n}\right| = \left|\frac{(n+1)!}{n!}\right| = n+1 \xrightarrow[n\to\infty]{} \infty = L$$

the radius is $R = \frac{1}{L} = \underline{\underline{0}}$

P.s. conv. at $x=0$ (center) only