Episode 33: Alternating series test

Any two consecutive term have opposite sign : $a_n \cdot a_{n+1} < 0$ for any $n = 1, 2, 3, \dots$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$$

Any alt. series can be written as

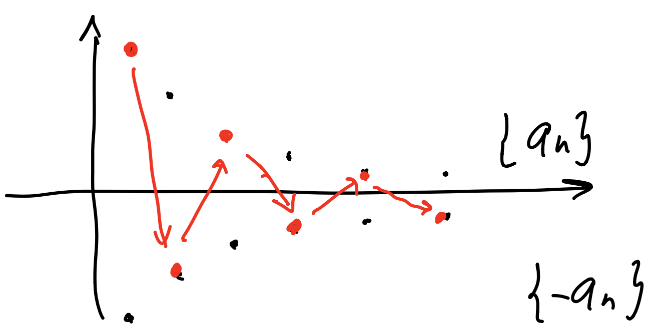
$$\left.\begin{aligned} &\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - \dots \\ &\qquad\text{or} \\ &\sum_{n=1}^{\infty} (-1)^{n} a_n = -a_1 + a_2 - a_3 + \dots \end{aligned}\right\} \text{ where } a_n > 0 \text{ for all } n$$

Alternating series test

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

1) $a_n > 0$ pos.
2) $a_{n+1} < a_n$ decreasing
3) $a_n \xrightarrow[n\to\infty]{} 0$

$\Rightarrow \sum_{1}^{\infty} (-1)^{n+1} a_n$ conv.



$\{a_n\}$ pos. decr. to 0 sequence

Ex. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \underbrace{\frac{1}{n}}_{a_n}$ cov/div?

alternating harmonic series

Apply alt. series test :

1) $a_n = \frac{1}{n} > 0$

2) $a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$ (decr.)

3) $a_n = \frac{1}{n} \xrightarrow[n\to\infty]{} 0$

$\Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ conv.

by alt. series test

$\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \ln 2\right.$ - prove later)

Def. $\sum_1^\infty a_n$ conv. absolutely if $\sum_1^\infty |a_n|$ conv.

$\sum_1^\infty a_n$ conv. conditionally if it conv., but not abs.

($\sum_1^\infty a_n$ conv., but $\sum_1^\infty |a_n|$ div.)

- If a series conv. abs. then it conv.

Ex 1) $\sum_1^\infty \frac{(-1)^{n+1}}{n}$ conv.

$\sum_1^\infty \left|\frac{(-1)^{n+1}}{n}\right| = \sum_1^\infty \frac{1}{n}$ div. (harm. series) $\Rightarrow$

$\sum \frac{(-1)^{n+1}}{n}$ conv. conditionally

2) $\sum_1^\infty \frac{(-1)^n}{n^2}$ conv. abs. since

$\sum_1^\infty \left|\frac{(-1)^n}{n^2}\right| = \sum_1^\infty \frac{1}{n^2}$ conv. as p-series with $p = 2 > 1$

$\Rightarrow \sum_1^\infty \frac{(-1)^n}{n^2}$ conv.

Ex. $\sum_{n=2}^\infty \frac{\cos(n\pi)}{\ln n}$ conv.? conv. abs.? div.?

$\cos(n\pi) = \begin{cases} 1, & n \text{ is even} \\ -1, & n \text{ is odd} \end{cases} = (-1)^n$

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} = \sum_{n=2}^{\infty} (-1)^n \underbrace{\frac{1}{\ln n}}_{a_n}$$

Apply alt. series test

1) $a_n = \frac{1}{\ln n} > 0 \quad (n \geq 2)$

2) $a_{n+1} = \frac{1}{\ln(n+1)} < \frac{1}{\ln n} = a_n$

$\ln(n+1) > \ln n$ since $\ln x$ is decr. f-n

3) $a_n = \frac{1}{\ln n} \xrightarrow[n\to\infty]{} 0$

$\Rightarrow$

$\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}$ conv. by alt. series test

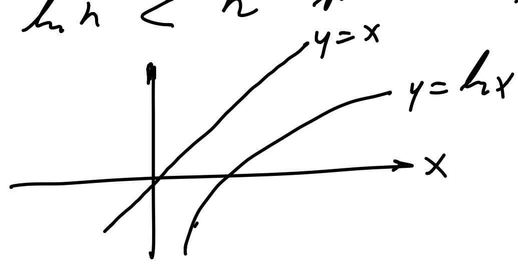
Does it conv. abs?

$$\left| \frac{(-1)^n}{\ln n} \right| = \frac{1}{\ln n}$$

$\sum_{n=2}^{\infty} \frac{1}{\ln n}$ compare with $\sum \frac{1}{n}$

pos.

$\ln n < n$ for all $n = 1, 2, 3$



$\frac{1}{\ln n} > \frac{1}{n}$

$\sum_{2}^{\infty} \frac{1}{n}$ div.

$\Leftarrow$

$\sum_{2}^{\infty} \frac{1}{\ln n}$ div. by comparison test

$$\sum_{2}^{\infty} \frac{(-1)^n}{\ln n} = \sum_{2}^{\infty} \frac{\cos(n\pi)}{\ln n}$$

conv., but not abs.
(conv. conditionally)