Episode 32: Ratio and root rests

Ratio test

$$\sum_{n=1}^{\infty} a_n \qquad \text{conv./div.?}$$

$$\left|\frac{a_{n+1}}{a_n}\right| \xrightarrow[n\to\infty]{} L$$

$L < 1 \Rightarrow$ the series <u>conv.</u>

$L > 1$ or $\infty \Rightarrow$ the series <u>div.</u>

$L = 1 \Rightarrow$ the test is inconclusive

<u>Ex.</u> $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$ con/div.?

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{(n+1)!} \div \frac{1}{n!} = \frac{n!}{(n+1)!} = \frac{1\cdot 2\cdot \ldots n}{1\cdot 2\cdot \ldots n\,(n+1)} = \frac{1}{n+1} \xrightarrow[n\to\infty]{} 0$$

$\underbrace{0}_{L} < 1 \Rightarrow$ the series conv. by the ratio test

<u>Ex.</u> $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$ conv./div.?

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{n+1}\cdot \cancel{n!}}{\cancel{(n+1)!}_{(n+1)}\cdot n^n} = \frac{(n+1)^{n+1}}{(n+1)\cdot n^n} = \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n =$$

$$= \left(1+\frac{1}{n}\right)^n \xrightarrow[n\to\infty]{(*)} \underbrace{e}_{L} > 1 \Rightarrow \text{the series div. by the ratio test}$$

$(*)$ $$\left(1+\frac{1}{n}\right)^n = e^{n\ln\left(1+\frac{1}{n}\right)} = e^{n\ln\left(1+\frac{1}{n}\right)} \xrightarrow[n\to\infty]{} e \qquad \left(n\ln\left(1+\frac{1}{n}\right) \xrightarrow[n\to\infty]{} 1\right)$$

$$\lim_{n\to\infty} n\cdot\ln\left(1+\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} \underset{\frac{1}{n}=x}{=} \lim_{x\to 0} \frac{\ln(1+x)}{x} \quad \left[\frac{0}{0}\right]$$

$$\underset{\text{L'Hop.}}{=} \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = 1$$

Root test

$\sum_{n=1}^{\infty} a_n$ conv / div. ?

$$\sqrt[n]{|a_n|} \xrightarrow[n\to\infty]{} L$$

$L<1 \Rightarrow$ the series conv.

$L>1$ or $\infty \Rightarrow$ the series div.

$L=1 \Rightarrow$ the test is inconclusive

Ex. $\sum_{n=1}^{\infty} \frac{2^n}{n^n}$ conv / div. ?

$$\sqrt[n]{|a_n|} = \sqrt[n]{\frac{2^n}{n^n}} = \frac{2}{n} \xrightarrow[n\to\infty]{} \underbrace{0}_{L} < 1 \Rightarrow \text{the series conv. by the root test}$$