Episode 31: Convergence tests: integral, comparison, limit comparison

Convergence tests for series:

1. Integral test
2. Comparison test
3. Limit comparison test
4. Ratio test
5. Root test
6. Alternating series test

(1–3) for positive series
(4–5) for any series
(6) for alt. series only

\+ Divergence test

Integral test (for positive series)

$$\sum_{n=1}^{\infty} \underbrace{a_n}_{\text{pos.}} \quad \text{and} \quad \int_N^{\infty} \underbrace{f(x)}\,dx \quad \text{conv. / div. simultaneously}$$

$$a_n = f(n)$$

$\left.\begin{array}{l}\cdot \text{ pos.}\\ \cdot \text{ cont.}\\ \cdot \text{ decreasing}\end{array}\right\}$ on $[N, \infty)$

Ex.

$$\boxed{\text{p-series} \quad \sum_{n=1}^{\infty} \frac{1}{n^p} \quad \begin{array}{l} \xrightarrow{p>1} \text{conv.} \\ \xrightarrow[p\le 1]{} \text{div.} \end{array} \qquad p \text{ is a given number}}$$

Why?

case 1 $\quad p \le 0$

$$\frac{1}{n^p} = n^{-p} \quad (-p \ge 0) \quad \underset{n\to\infty}{\not\longrightarrow} 0 \implies \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ div.}$$

by the div. test

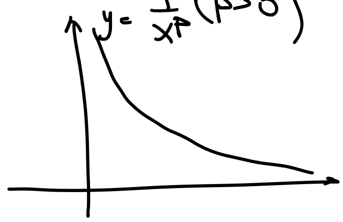
case 2 $\quad p > 0$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{conv / div. together with} \quad \int_1^{\infty} \frac{1}{x^p}\,dx \quad \begin{array}{l} \xrightarrow{p>1} \text{conv.} \\ \xrightarrow[p\le 1]{} \text{div.} \end{array}$$

pos.
cont.
decreasing

(p-integral)

$y = \frac{1}{x^p}\ (p>0)$



For example, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ conv. as p-series with $p = 2 > 1$

$\left(\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \right)$ $\left(\sum_{n=1}^{\infty} \frac{1}{n^3} \quad \text{conv.} \right)$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ div. as p-series with $p = 1/2 \leq 1$

Comparison test (for pos. series)

$\sum_1^{\infty} a_n$, $\sum_1^{\infty} b_n$ pos series

1) $a_n \leq b_n$ for all n, $\sum_{n=1}^{\infty} b_n$ conv. $\Rightarrow \sum_1^{\infty} a_n$ conv.

2) $a_n \leq b_n$ for all n, $\sum_{n=1}^{\infty} a_n$ div. $\Rightarrow \sum_{n=1}^{\infty} b_n$ div.

Model series to compare with:

- geom. $\sum_{n=1}^{\infty} ar^n$ $\xrightarrow{|r|<1}$ conv. ; $\xrightarrow[|r| \geq 1]{}$ div.

- p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $\xrightarrow{p>1}$ conv. ; $\xrightarrow[p \leq 1]{}$ div.

Ex. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ conv / div ?

$\underbrace{\frac{1}{n^2+1}}_{\text{pos.}} < \underbrace{\frac{1}{n^2}}_{\text{pos.}}$ for all n

$\sum_1^{\infty} \frac{1}{n^2}$ conv. as p-series with $p = 2 > 1$ $\Rightarrow$

$\sum_1^{\infty} \frac{1}{n^2+1}$ conv. by comparison test

Limit comparison (for pos. series)

$\sum_1^\infty a_n$, $\sum_1^\infty b_n$

given , model series to compare with

$$\frac{a_n}{b_n} \xrightarrow[n\to\infty]{} L \begin{matrix} \neq 0 \\ \neq \infty \end{matrix} \quad \Bigg| \Longrightarrow \sum_1^\infty a_n , \sum_1^\infty b_n \text{ conv / div simultaneously}$$

Ex. $\sum_{n=1}^\infty \underbrace{\frac{1}{3^n - 2^n}}_{a_n}$ conv / div?

compare with $\sum_{n=1}^\infty \underbrace{\frac{1}{3^n}}_{b_n}$

$$\frac{a_n}{b_n} = \frac{1}{3^n - 2^n} \div \frac{1}{3^n} = \frac{3^n}{3^n - 2^n} = \frac{1}{1 - \left(\frac{2}{3}\right)^n} \xrightarrow[n\to\infty]{} 1 \begin{pmatrix} \neq 0 \\ \neq \infty \end{pmatrix}$$

$\left(\frac{2}{3}\right)^n \to 0$

$\Rightarrow \sum_{n=1}^\infty \frac{1}{3^n - 2^n}$ conv. since $\underbrace{\sum_{n=1}^\infty \frac{1}{3^n} \text{ conv.}}_{\text{geom series with } r = \frac{1}{3}}$

$|\frac{1}{3}| < 1$

by Limit comparison test