Episode 30: Divergence test and other theorems about series

A series $\sum_{n=1}^{\infty} a_n$ conv. / div. if the sequence of its partial sums $\{s_n\}_{n=1}^{\infty}$ conv. / div.

$$s_n = a_1 + \dots + a_n$$

<u>Th 1</u> (necessary condition for conv.)

$$\sum_{n=1}^{\infty} a_n \text{ conv.} \Rightarrow a_n \xrightarrow[n\to\infty]{} 0$$

<u>Proof</u> Let $\sum_{n=1}^{\infty} a_n = S$. This means that

$$s_n \xrightarrow[n\to\infty]{} S$$

$$\begin{array}{rl} & s_n = a_1 + \dots + a_{n-1} + a_n \\ - & s_{n-1} = a_1 + \dots + a_{n-1} \\ \hline & a_n = \underbrace{s_n}_{\xrightarrow[n\to\infty]{} S} - \underbrace{s_{n-1}}_{\xrightarrow[n\to\infty]{} S} \xrightarrow[n\to\infty]{} S - S = \underline{0} \end{array}$$

as required

Equivalent formulation:

$$\boxed{\text{If } a_n \underset{n\to\infty}{\not\to} 0 \text{ then } \sum_{n=1}^{\infty} a_n \text{ div.}}$$

divergence test

Warning: $a_n \xrightarrow[n\to\infty]{} 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ conv.

Wrong

For harm. series $\sum_{n=1}^{\infty} \frac{1}{n}$, $a_n = \frac{1}{n} \xrightarrow[n\to\infty]{} 0$, but

$$\sum_{n=1}^{\infty} \frac{1}{n} \ \underline{\text{div.}}$$

<u>Ex.</u> $\sum_{n=1}^{\infty} \sqrt[n]{2} = 2 + \sqrt{2} + \sqrt[3]{2} + \sqrt[4]{2} + \dots$

conv / div ?

$$a_n = \sqrt[n]{2} = 2^{\left(\frac{1}{n}\right) \to 0} \xrightarrow[n\to\infty]{} 2^0 = 1 \neq 0 \Rightarrow$$

$\sum_{n=1}^{\infty} \sqrt[n]{2}$ div by the divergence test

Th 3 (sum of series)

If $\sum_1^{\infty} a_n$, $\sum_1^{\infty} b_n$ are conv. then

$\sum_{n=1}^{\infty} \underbrace{c}_{\text{const}} a_n$, $\sum_{n=1}^{\infty} (a_n \pm b_n)$ are conv. and

$$\| \qquad\qquad \|$$

$$c \sum_{n=1}^{\infty} a_n \qquad \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$$

Ex. 1 $\sum_{n=1}^{\infty} \frac{1+2^{n+1}}{3^n}$ conv. / div ?

$$\frac{1+2^{n+1}}{3^n} = \left(\frac{1}{3}\right)^n + 2\left(\frac{2}{3}\right)^n$$

$$\underbrace{\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n}_{\substack{\text{conv. series} \\ \text{(as geom series} \\ \text{with } r=\frac{1}{3} \\ |\frac{1}{3}|<1)}} + 2\underbrace{\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n}_{\substack{\text{conv. series} \\ \text{as geom.} \\ \text{with } r=\frac{2}{3} \\ |\frac{2}{3}|<1}}$$

$$\sum_{n=1}^{\infty} \frac{1+2^{n+1}}{3^n} = \underbrace{\sum_1^{\infty}\left(\frac{1}{3}\right)^n}_{} + 2\underbrace{\sum_1^{\infty}\left(\frac{2}{3}\right)^n}_{} =$$

conv.

$$= \frac{\frac{1}{3}}{1-\frac{1}{3}} + 2\cdot\frac{\frac{2}{3}}{1-\frac{2}{3}} = \frac{1}{2} + 2\cdot\frac{2}{1} = \boxed{\frac{9}{2}}$$

So $\sum_{1} \frac{1+2^{n+1}}{3^n}$ conv. to $\frac{9}{2}$

Ex. 2 $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{2^n}\right)$ conv. / div. ?

Assume that $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{2^n}\right)$ conv. Then

was wrong

$$\underbrace{\underbrace{\sum_{1}^{\infty} \left(\frac{1}{n} - \frac{1}{2^n}\right)}_{\text{conv. by assumption}} + \underbrace{\sum \frac{1}{2^n}}_{\text{conv. (as geom.)}}}_{\text{conv. (by Thm 2)}} = \sum_{1}^{n} \left(\frac{1}{n} - \frac{1}{2^n} + \frac{1}{2^n}\right) = \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\text{diverges}} \text{ harm. series}$$

contradiction!

So the assumption was wrong and $\sum_{1}^{\infty} \left(\frac{1}{n} - \frac{1}{2^n}\right)$ div.

Remark

$$\underbrace{\sum_{1}^{\infty} \left(\frac{1}{n} - \frac{1}{2^n}\right)}_{\text{div.}} \underset{\text{wrong}}{=} \underbrace{\sum_{1}^{\infty} \frac{1}{n}}_{\text{div.}} - \sum_{1}^{\infty} \frac{1}{2^n}$$

Similarly,

$$\sum_{1}^{\infty} \frac{1}{n(n+1)} = \underbrace{\sum_{1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)} \overset{\text{wrong}}{=} \underbrace{\sum_{1}^{\infty} \frac{1}{n}}_{\text{div.}} - \underbrace{\sum_{1}^{\infty} \frac{1}{n+1}}_{\text{div.}}$$

(telescoping series with sum 1)

Conv.