Episode 29: Series

$a_1, a_2, a_3, \ldots, a_n, \ldots$ sequence $\{a_n\}_{n=1}^{\infty}$

$a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ series $\sum_{n=1}^{\infty} a_n$

What is the infinite sum $\sum_{n=1}^{\infty} a_n$?

Do summation step by step

partial sums (finite)
$$\begin{cases} S_1 = a_1 \\ S_2 = a_1 + a_2 \\ S_3 = a_1 + a_2 + a_3 \\ \ldots \\ S_n = a_1 + a_2 + \ldots + a_n = \sum_{i=1}^{n} a_i \\ \ldots \end{cases}$$

Consider a sequence $\{S_1, S_2, \ldots, S_n, \ldots\} = \{S_n\}_{n=1}^{\infty}$

If this sequence of partial sums converges to limit s ($\lim_{n\to\infty} S_n = s$) then we say that the series $\sum_{n=1}^{\infty} a_n$ converges to s and write

$$\sum_{n=1}^{\infty} a_n = S.$$

If the seq. of partial sums diverges ($\lim_{n\to\infty} S_n$ DNE), then we say that the series $\sum_{n=1}^{\infty} a_n$ diverges

Ex. $\sum_{n=1}^{\infty} (-1)^n = \underset{n=1}{-1} + \underset{n=2}{1} - 1 + 1 - \ldots$ conv. or div.?

Partial sums

$S_1 = -1$

$S_2 = -1 + 1 = 0$

$S_3 = -1 + 1 - 1 = -1$

$S_4 = 0$

$\{S_n\}_{n=1}^{\infty} = \{-1, 0, -1, 0, \ldots\}$ diverges $\Rightarrow \sum_{n=1}^{\infty} (-1)^n$ div.

Geom. series

$1 + r + r^2 + r^3 + \dots = \sum_{n=1}^{\infty} r^{n-1} \quad \left(= \sum_{n=0}^{\infty} r^n \right)$ conv. or div.?

case 1 $r = 1$

$1 + 1 + 1 + \dots = \sum_{n=1}^{\infty} 1^n$

Seq. of partial sums:

$s_1 = 1$

$s_2 = 2$

$s_3 = 3$

$\dots$

$s_n = n$

$\{s_n\}_{n=1}^{\infty} = \{n\}_{n=1}^{\infty}$ div. to ∞ $\Rightarrow$ geom. series with $r=1$ div. to ∞

case 2 $r \neq 1$

$$s_n = 1 + r + r^2 + \dots + r^{n-1}$$

$$- \quad r s_n = r + r^2 + r^3 + \dots + r^n$$

$$\underbrace{s_n - r s_n}_{s_n(1-r)} = 1 - r^n \underset{r \neq 1}{\Rightarrow} s_n = \frac{1-r^n}{1-r} \xrightarrow[n\to\infty]{} ?$$

As we know, $\lim_{n\to\infty} r^n = \begin{cases} 1, & r = 1 \quad \times \\ 0, & -1 < r < 1 \\ \text{DNE}, & r \leq -1 \text{ or } r > 1 \end{cases}$

So $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1 - r^n}{1-r} = \begin{cases} \frac{1}{1-r}, & -1 < r < 1 \\ \text{DNE}, & r \leq -1 \text{ or } r > 1 \end{cases}$.

case 1 + case 2:

$$\sum_{n=1}^{\infty} r^n = \lim_{n\to\infty} s_n = \begin{cases} \frac{1}{1-r}, & -1 < r < 1 \quad (|r| < 1) \\ \text{DNE}, & \text{otherwise} \quad (|r| \geq 1) \end{cases}$$

$$1 + r + r^2 + \dots \underset{|r|<1}{=} \frac{1}{1-r} \quad \Bigg\} \quad 1 + x + x^2 + \dots \underset{|x|<1}{=} \frac{1}{1-x}$$

1^{st} term

Similary

$$\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \dfrac{a}{1-r}, & |r|<1 \\ \text{diverges}, & |r| \geq 1 \end{cases}$$

1st term

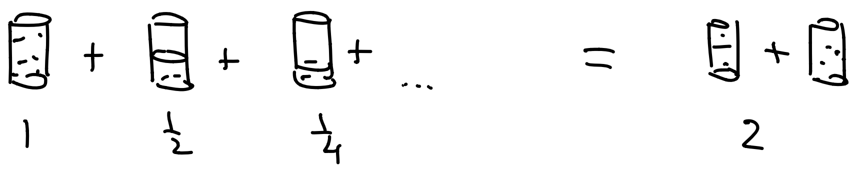
common radio

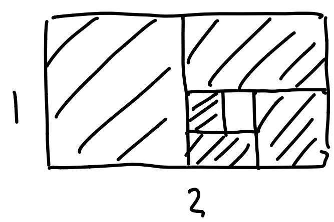
Ex. $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$

1st term

common ratio

geom. series with
$r=\frac{1}{2}$
$|\frac{1}{2}|<1 \Rightarrow$ the series conv.



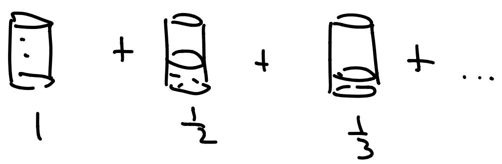


$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots = 2$

Harmonic series

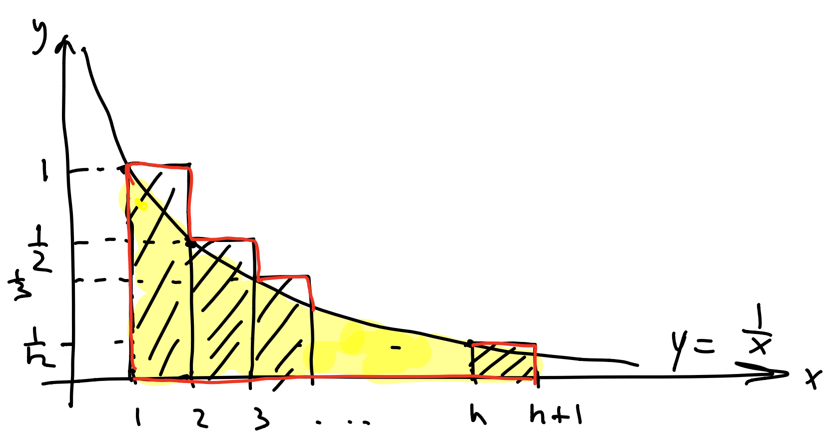
$$\sum_{n=1}^{\infty}\frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$$

conv. / div.?

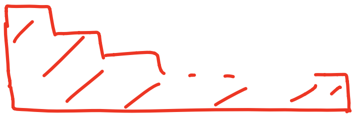


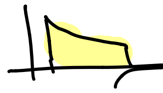
∞

$$S_n = 1+\frac{1}{2}+\ldots+\frac{1}{n} \xrightarrow[n\to\infty]{} ?$$



$$S_n = 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n} > \int_1^{n+1}\frac{1}{x}\,dx = \ln x\Big|_1^{n+1} = \ln(n+1) \xrightarrow[n\to\infty]{} \infty$$





So $\{S_n\}_{n=1}^{\infty}$ div. to $\infty \Rightarrow \boxed{\sum_{n=1}^{\infty}\frac{1}{n} \text{ div. to } \infty}$

Telescoping series

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\dots+\frac{1}{n(n+1)}+\dots$$

conv / div. ?

$$S_n = \sum_{i=1}^{n}\frac{1}{i(i+1)} \underset{\substack{\text{partial}\\\text{fractions}\\\text{decom.}}}{=} \sum_{i=1}^{n}\left(\frac{1}{i}-\frac{1}{i+1}\right) =$$

$$= \underbrace{\left(\frac{1}{1}-\cancel{\frac{1}{2}}\right)}_{i=1} + \underbrace{\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)}_{i=2} + \underbrace{\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)}_{i=3} + \dots + \underbrace{\left(\cancel{\frac{1}{n}}-\frac{1}{n+1}\right)}_{i=n} =$$

$$= \underbrace{1-\frac{1}{n+1}}_{S_n} \xrightarrow[n\to\infty]{} 1$$

$$S_n \xrightarrow[n\to\infty]{} 1 \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1$$