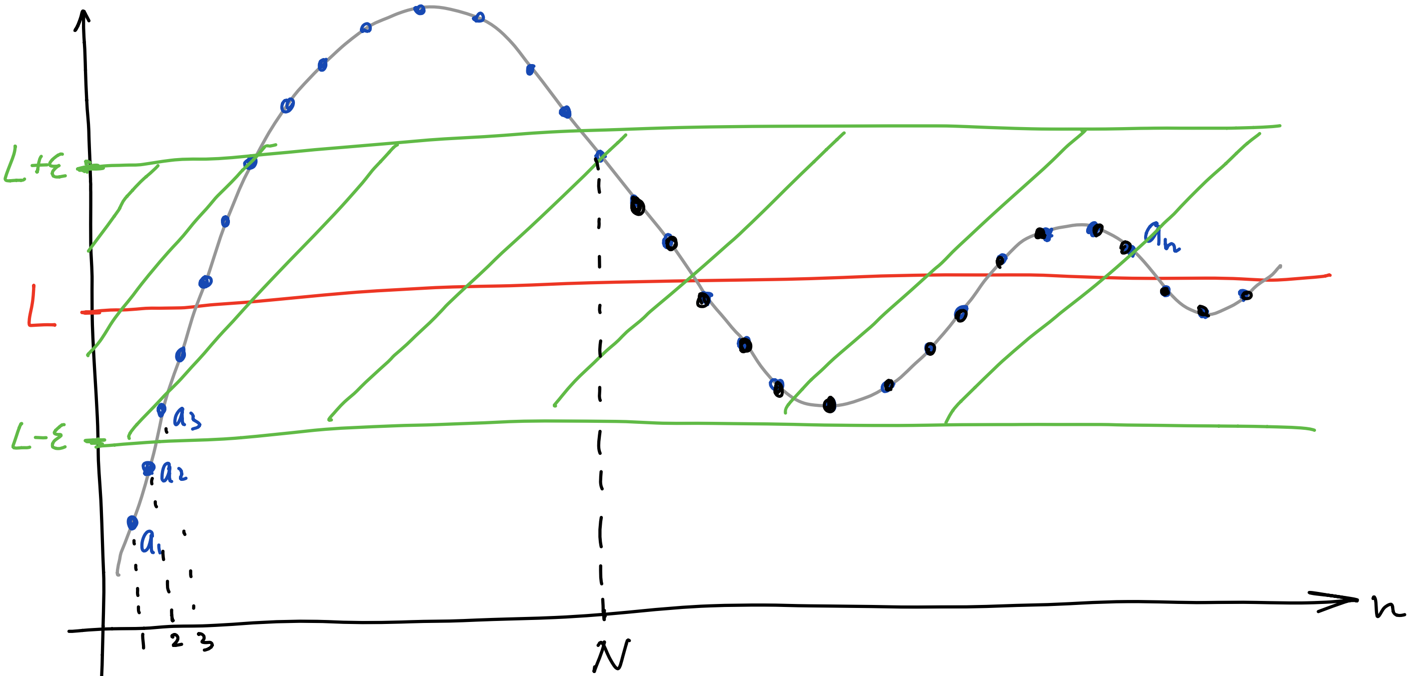
Episode 28. Limit of a sequence

Definition. A real number L is called *limit* of a sequence $\{a_n\}_{n=1}^{\infty}$ if for any positive number ε there exists a number N such that $|a_n - L| < \varepsilon$ whenever $n > N$.

Notation: $\lim\limits_{n\to\infty} a_n = L$ or $a_n \underset{n\to\infty}{\longrightarrow} L$.



$$n > N \Rightarrow |a_n - L| < \varepsilon$$

$$\Leftrightarrow L - \varepsilon < a_n < L + \varepsilon$$

$\lim\limits_{n\to\infty} a_n = L$ if a_n becomes arbitrary close to L for suff. large n.

$\{a_n\}_{n=1}^{\infty}$ $\xrightarrow{\text{converges}}$ if there exist $\lim\limits_{n\to\infty} a_n = L < \infty$

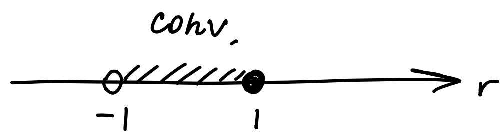
$\{a_n\}_{n=1}^{\infty}$ $\xrightarrow[\text{diverges}]{}$ if $\lim\limits_{n\to\infty} a_n$ DNE

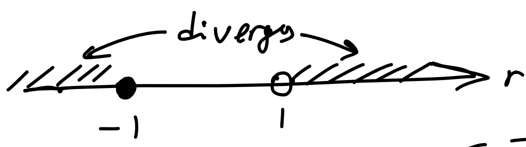
Ex. 1 Harmonic seq. $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ conv. to 0

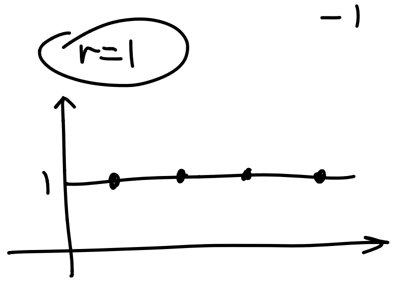
means $a_n = \frac{1}{n} \underset{n\to\infty}{\longrightarrow} 0$

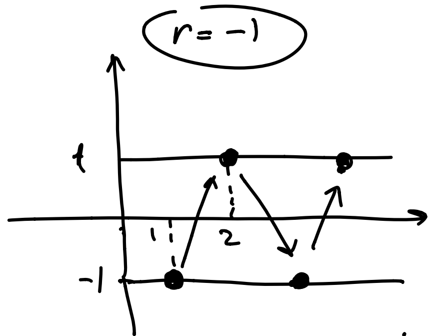
Ex. 2 Geometric seq. $\{r, r^2, r^3, \dots\} = \{r^n\}_{n=1}^{\infty}$

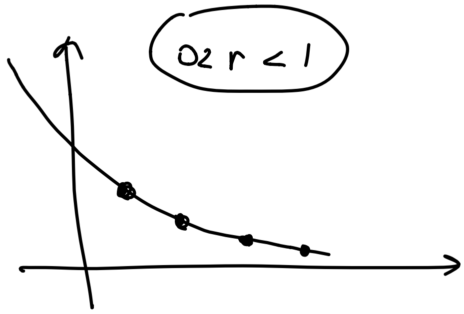
$$\lim_{n\to\infty} r^n = \begin{cases} 1, & r = 1 \\ 0, & -1 < r < 1 \\ \text{DNE}, & r \le -1 \text{ or } r > 1 \end{cases}$$

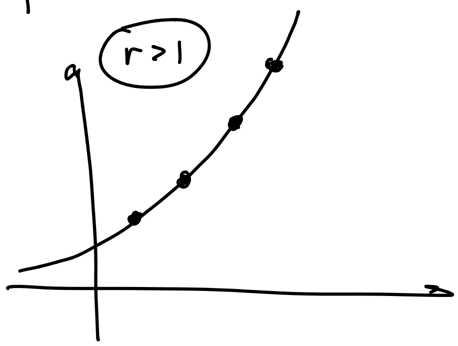


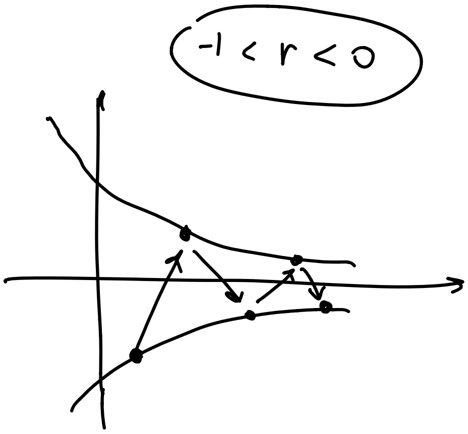


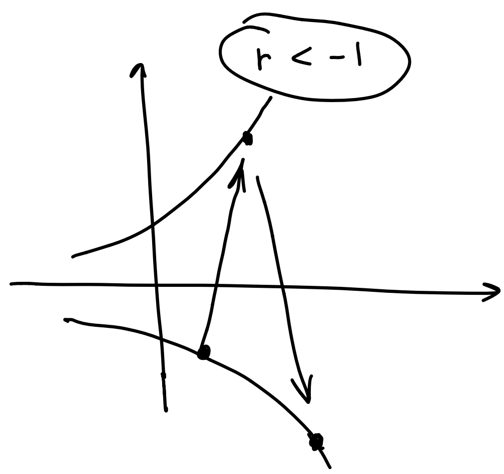












Properties of the limit

1. lim is unique (if exists)
2. lim of $+, -, \times, \div$, power of limits

is equal to the $+, -, \times, \div$, power of limits (if exist)

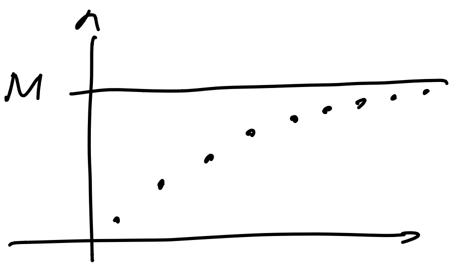
$$\left(\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n \right)$$

3. squeeze th

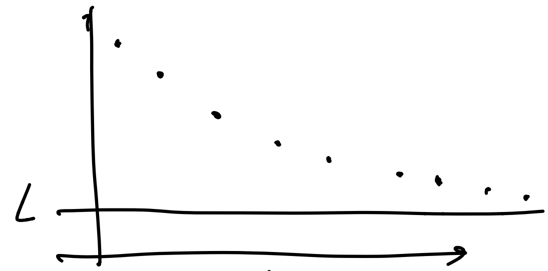
$$b_n \le a_n \le c_n$$

$b_n \xrightarrow{n\to\infty} L$, $a_n \xrightarrow{n\to\infty} L$, $c_n \xrightarrow{n\to\infty} L$

4. A seq. converges if its extension f-n converges
5. <u>Bounded</u> <u>monotonic</u> sequences converge



incr. + bounded above



decr. + bounded below

Comparative asymptotic behavior of sequences

at ∞

log	power	exp	factorial
$\ln n$	n^2	2^n	$n! = 1\cdot 2\cdot 3 \cdot \ldots \cdot n$ (has no extension f-n)
$\log_2 n$	$n^{1/3}$	e^n	
$\vdots$	$\vdots$	$\vdots$	
$\log_a n \ (a>1)$	$n^a \ (a>0)$	$a^n \ (a>1)$	$n!$
$\downarrow n\to\infty$	$\downarrow n\to\infty$	$\downarrow n\to\infty$	$\downarrow n\to\infty$
∞	∞	∞	∞

Which sequence does grow faster?

$$\log_a n < n^a < a^n < n!$$

log power exp factorial

How to compare the growth at ∞?

Ex.

(1)

$$\lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} \text{ log, power} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty} \frac{\ln x}{x^{\frac{1}{2}}} \underset{\text{L'Hop}}{=}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2}x^{-\frac{1}{2}}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = \underline{0} \Rightarrow$$ ln grows faster at ∞ than power

In general, $\lim_{n\to\infty} \frac{\log_a n}{n^b} = 0$ for any $a > 1$, $b > 0$

$$\log_a n < n^b \quad \text{as } n \to \infty$$

(2)

$$\lim_{n\to\infty} \frac{n^2}{2^n} \text{ power, exp.} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty} \frac{x^2}{2^x} \underset{\text{L'Hop}}{=} \lim_{x\to\infty} \frac{2x}{\ln 2 \cdot 2^x} =$$

$$\left[\frac{\infty}{\infty}\right] \underset{\text{L'H}}{=} \lim_{x\to\infty} \frac{2}{(\ln 2)^2 2^x} = \underline{0} \Rightarrow$$ power grows faster at ∞ than exp

In general, $\frac{n^a}{b^n} \xrightarrow[n\to\infty]{} 0$ for any $a > 0$, $b > 1$

(3) 2^n exp

like $n!$ factorial

$n \to \infty$

$$\frac{2^n}{n!} = \underbrace{\frac{2}{1}}_{2} \cdot \underbrace{\frac{2}{2}}_{1} \cdot \underbrace{\frac{2}{3}}_{<1} \cdot \underbrace{\frac{2}{4}}_{<1} \cdots \cdots \underbrace{\frac{2}{(n-1)}}_{<1} \cdot \underbrace{\frac{2}{n}}_{= \frac{2}{n}}$$

$$< 2 \cdot 1 \cdot 1 \cdot 1 \cdots \cdot 1 \cdot \frac{2}{n} = \frac{4}{n}$$

$$0 < \frac{2^n}{n!} < \frac{4}{n}$$

$n \to \infty$ $n \to \infty$ $n \to \infty$

0

by the squeeze th

So $\frac{2^n}{n!} \xrightarrow[n \to \infty]{} 0$ the factorial grows faster than the exp

In general, $\frac{a^n}{n!} \xrightarrow[n \to \infty]{} 0$ for any $a > 1$