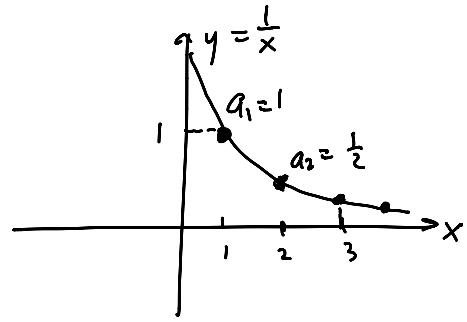
Episode 27: Model sequences

① Harmonic sequence

$$\left\{\frac{1}{n}\right\}_{n=1}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$$

Extension f-n is $f(x) = \frac{1}{x}$



② Geometric sequence

Given numbers

a (first term)

r (common ratio)

$$\{a, ar, ar^2, ar^3, \dots, ar^{n-1}, \dots\} = \{\underbrace{ar^{n-1}}_{\text{general term}}\}_{n=1}^{\infty}$$

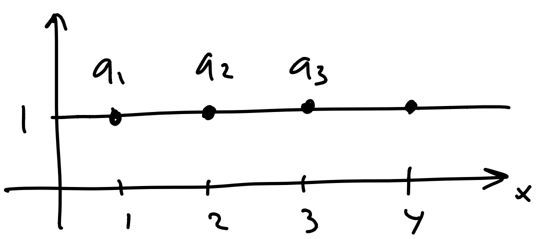
General term $a_n = a \cdot r^{n-1}$

Recursive formula $\begin{cases} a_1 = a \\ a_{n+1} = a_n \cdot r \end{cases} \quad n = 1, 2, \dots$

$$r = \frac{a_{n+1}}{a_n} = \frac{a_n}{a_{n-1}} = \dots = \frac{a_2}{a_1}$$

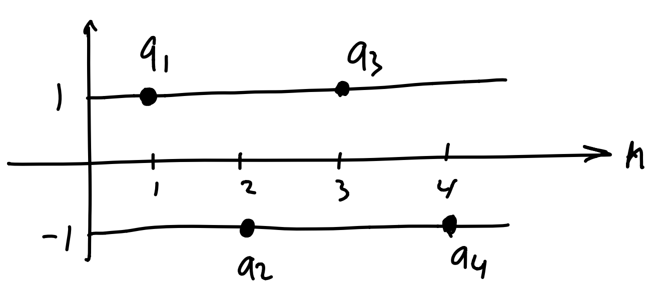
Ex.

1) $a = 1, \ r = 1$ $\quad \{1, 1, 1, \dots\}$ $\quad a_n = 1$ for all n



2) $a = 1, \ r = -1$ $\quad \{1, -1, 1, -1, \dots\}$

$a_n = (-1)^{n+1}$

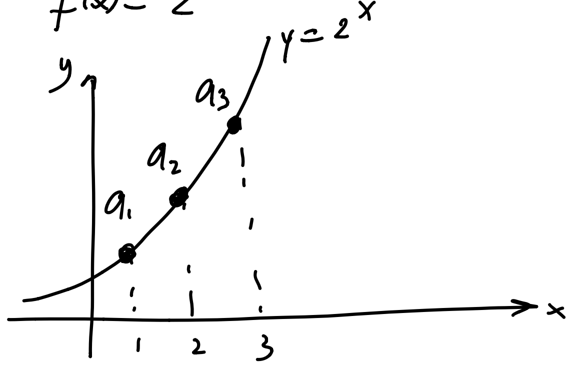


3) $\{2^n\}_{n=1}^{\infty} = \{2, 4, 8, 16, \dots\}$

$a = 2$

$r = \frac{4}{2} = 2$

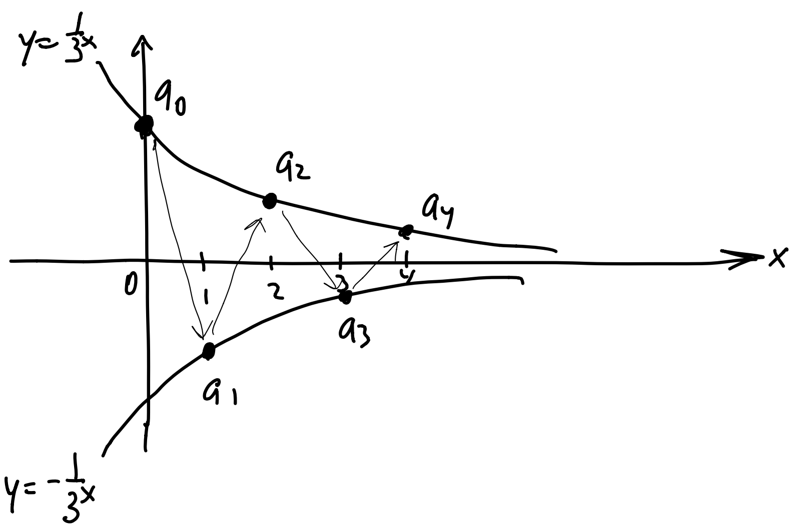
$f(x) = 2^x$



4) $\{\underbrace{(-\frac{1}{3})^n}_{\text{formula for gen. term}}\}_{n=0}^{\infty} = \{\underset{a_0}{1}, \underset{a_1}{-\frac{1}{3}}, \underset{a_2}{\frac{1}{9}}, -\frac{1}{27}, \dots\}$

$a = 1$

$r = -\frac{1}{3}$



Terminology

A seq. $\{a_n\}_{n=1}^{\infty}$ is called

increasing if $a_n < a_{n+1}$ for all n

decreasing if $a_n > a_{n+1}$ — " —

monotonic if it's either incr. or decr.

positive if $a_n > 0$ for all n

negative if $a_n < 0$ — " —

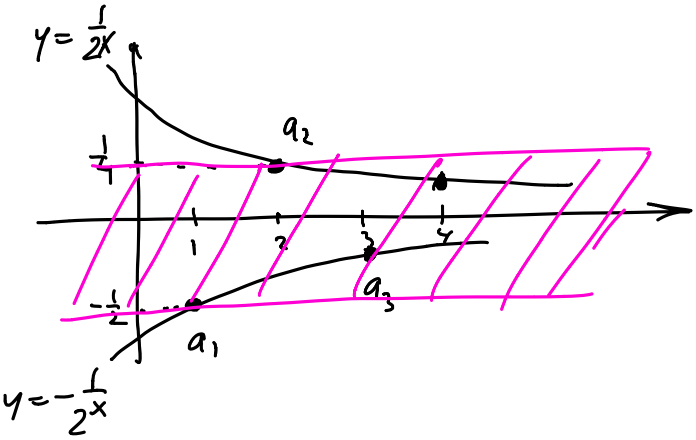
alternating if $a_n a_{n+1} < 0$ for all n

bounded above if $a_n \le M$ all n for some const $\underbrace{M}_{\text{upper bound}}$

bounded below if $a_n \geq L$ for all n for some const $\underbrace{L}_{\text{lower bound}}$

EX

1) $\left\{ \left(-\frac{1}{2}\right)^n \right\}_{n=1}^{\infty} = \left\{ -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \dots \right\}$ geom. seq.



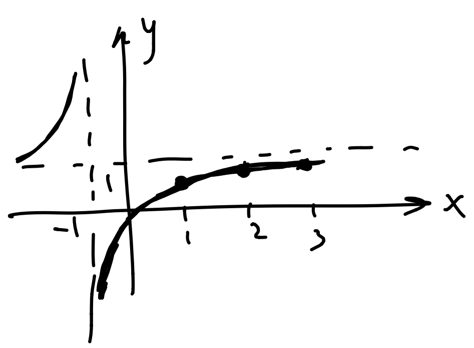
- neither incr. nor decr. (so not monotonic)
- alternating
- bounded above $\frac{1}{4}$ $(= a_2)$
- bounded below by $-\frac{1}{2}$ $(= a_1)$

2) $\{3^n\}_{n=1}^{\infty}$ increasing
positive
bounded below by 3

3) $\{a_n\}_{n=1}^{\infty}$ $a_n = \frac{n}{n+1}$

positive

Extension f-n $f(x) = \frac{x}{x+1} = \frac{(x+1)-1}{x+1} = 1 - \frac{1}{x+1}$



$\{a_n\}_1^{\infty}$ incr.
bounded above by $\underline{1}$

OR $f'(x) = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2} > 0$

$\Rightarrow f$ is $\underline{\text{incr.}}$

Alternative sol.

$$a_n = \frac{n}{n+1} = \frac{(n+1)-1}{n+1} = 1 - \frac{1}{n+1} < 1 \quad \text{for all } n$$

$\frac{1}{n+1} > 0$

1 upper bound for $\{a_n\}_{n=1}^{\infty}$

$$a_{n+1} = \frac{n+1}{n+2} = 1 - \frac{1}{n+2} > 1 - \frac{1}{n+1} = a_n \quad \text{for all } n$$

so $a_{n+1} > a_n \Rightarrow \{a_n\}$ is incr.