Episode 25

Second-order differential equations

Objectives

In this episode, we will learn how to solve

second-order homogeneous linear differential equation with constant coefficients.

What this long name means? A general equation of this type is written as

$$a\,\frac{d^2y}{dx^2} + b\,\frac{d\,y}{dx} + cy = 0, \quad \text{or} \quad ay'' + by' + cy = 0\,,$$

where a, b, c are given constants (they are called constant coefficients)
and $y = y(x)$ is unknown function in variable x.

Second-order means that the highest derivative is of the second order.
For this reason, $a \neq 0$.

Homogeneous means that the right hand side of the equation is zero.

Linear means that y, y' and y'' are involved with exponents of one.

For example, $3y'' + y' - 2y = 0$ and $y'' = 0$ are
second-order homogeneous linear equation with constant coefficients,

while $y'' - y^2 = 0, \quad y'' + 2y' = x + 1\,, \quad y' + y = 0\,, \quad y'' - xy' = 0$ are not.

Solutions and their linear behavior

A *solution* of the equation $ay'' + by' + cy = 0$ is any function satisfying the equation.
For example, $y = e^x$ is a solution of the equation $y'' - y = 0$, because $(e^x)'' - e^x = 0$.
This solution is not unique: $y = e^{-x}$ is also a solution since $(e^{-x})'' - e^{-x} = 0.$
Actually, the equation $y'' - y = 0,$ as any other second-order linear equation,
has infinitely many solutions. How to find them **all**?

Theorem (linear behavior of solutions). If y_1 and y_2 are solutions of $ay'' + by' + cy = 0,$
then so $C_1y_1 + C_2y_2$ is for any constants C_1 and C_2.

Proof. Let $y_1,\ y_2$ be solutions of $ay'' + by' + cy = 0.$ Then $ay_1'' + by_1' + cy_1 = 0$ and $ay_2'' + by_2' + cy_2 = 0.$
Let us check that $C_1y_1 + C_2y_2$ satisfies the equation for any choice of constants $C_1,\ C_2$:

$a(C_1y_1 + C_2y_2)'' + b(C_1y_1 + C_2y_2)' + c(C_1y_1 + C_2y_2) \stackrel{?}{=} 0$

$aC_1y_1'' + aC_2y_2'' + bC_1y_1' + bC_2y_2' + cC_1y_1 + cC_2y_2 \stackrel{?}{=} 0$

$C_1(ay_1'' + by_1' + cy_1) + C_2(ay_2'' + by_2' + cy_2) \stackrel{?}{=} 0$

$C_1 \cdot 0 + C_2 \cdot 0 \stackrel{?}{=} 0$

$0 = 0 \quad \checkmark$

Proportional solutions

Example 1. As we saw, $y_1 = e^x$ and $y_2 = e^{-x}$ are solutions of $y'' - y = 0$.
Therefore, $y = C_1 e^x + C_2 e^{-x}$ is also a solution for any $C_1, C_2 \in \mathbb{R}$.
Are there any other solutions besides $y = C_1 e^x + C_2 e^{-x}$? (Spoiler: no!)

Example 2. The equation $y'' + y = 0$ has solutions $y_1 = \sin x$ and $y_2 = 5 \sin x$. Indeed,
$y_1'' = (\sin x)'' = (\cos x)' = -\sin x$, so $y_1'' + y_1 = -\sin x + \sin x = 0$,
and $y_2'' = (5 \sin x)'' = (5 \cos x)' = -5 \sin x$, so $y_2'' + y_2 = -5 \sin x + 5 \sin x = 0$.
Therefore, $y = C_1 \sin x + C_2 \cdot 5 \sin x = (C_1 + 5C_2) \sin x = C \sin x$ is a solution for any constant C.
Are here any other solutions besides $y = C \sin x$?
(Spoiler: yes! Take, for example, $y = \cos x$.)

What is the difference between these two pairs of solutions,
$y_1 = e^x$ and $y_2 = e^{-x}$ in Example 1 and $y_1 = \sin x$ and $y_2 = 5 \sin x$ in Example 2?

It's easy to observe that

$\sin x$ and $5 \sin x$ are **proportional**, while e^x and e^{-x} are not.

Indeed, $y_1 = \sin x$ and $y_2 = 5 \sin x$ are proportional since $y_2 = 5y_1$.
If we assume that e^x and e^{-x} are proportional, then $y_2 = Cy_1$ for some constant C.
But $e^{-x} = Ce^x \iff 1 = Ce^{2x}$ for all x, which is impossible.

General solution

Two proportional solutions are called *linearly dependent*.
Non-proportional solutions are called *linearly independent*.

$y_1 = e^x$ and $y_2 = e^{-x}$ are *linearly independent* solutions of $y'' - y = 0$.

$y_1 = \sin x$ and $y_2 = 5 \sin x$ are *linearly dependent* solutions of $y'' + y = 0$.

Definition. A *general solution* of the equation $ay'' + by' + cy = 0$ is

$$y(x) = C_1 y_1(x) + C_2 y_2(x),$$

where $y_1(x),\ y_2(x)$ are linearly independent solutions of the equation
and $C_1,\ C_2$ are arbitrary constants.
The expression $C_1 y_1(x) + C_2 y_2(x),$ is called a *linear combination* of y_1 and y_2.

As it will be proven in the course of differential equations,
any solution of the equation $ay'' + by' + cy = 0$ can be obtained from the general solution by an appropriate choice of the constants $C_1,\ C_2$.

Therefore, to find a general solution, we need to find two linearly independent solutions y_1, y_2 and set up their linear combination $C_1 y_1 + C_2 y_2$ with arbitrary constants $C_1,\ C_2$.

How to find linearly independent solutions

We know that the first-order equation $ay' + by = 0$ has an exponential solution (it can be found by separation of variables).

Let us try to find a solution of $ay'' + by' + cy = 0$ in the exponential form $y = e^{\lambda x}$, where λ is unknown constant (to be determined).
Substitute $y = e^{\lambda x}$, $y' = \lambda e^{\lambda x}$, and $y'' = \lambda^2 e^{\lambda x}$ into the equation:

$$a\underbrace{\lambda^2 e^{\lambda x}}_{y''} + b\underbrace{\lambda e^{\lambda x}}_{y'} + c\underbrace{e^{\lambda x}}_{y} = 0.$$

Factor out $e^{\lambda x}$:

$a\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} = 0 \iff e^{\lambda x}(a\lambda^2 + b\lambda + c) = 0$

We may cancel $e^{\lambda x}$ out since it's never zero:
$e^{\lambda x}(a\lambda^2 + b\lambda + c) = 0 \iff a\lambda^2 + b\lambda + c = 0.$

If $y = e^{\lambda x}$ is a solution of the equation $ay'' + by' + cy = 0,$
then λ is a root of the quadratic equation $a\lambda^2 + b\lambda + c = 0$.

The quadratic equation $a\lambda^2 + b\lambda + c = 0$ is called
the *characteristic equation* of the differential equation $ay'' + by' + cy = 0.$

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The characteristic equation: real roots

The solutions of the characteristic equation are $\lambda_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$

• If λ_1, λ_2 are real and distinct, then two linearly independent solutions are $y_1(x) = e^{\lambda_1 x}$ and $y_2(x) = e^{\lambda_2 x}$.

• If $\lambda_1 = \lambda_2\,(= \lambda = -b/2a)$ is a double root, then two linearly independent solutions are $y_1(x) = e^{\lambda x}$ and $y_2(x) = xe^{\lambda x}$.

Let us check that $y_2(x) = xe^{\lambda x}$ is a solution of $ay'' + by' + cy = 0.$
Substitute $y_2'(x) = (1 + \lambda x)e^{\lambda x}$ and $y_2''(x) = (2\lambda + \lambda^2 x)e^{\lambda x}$
into the left hand side of the equation:
$a(2\lambda + \lambda^2 x)e^{\lambda x} + b(1 + \lambda x)e^{\lambda x} + cxe^{\lambda x} = (a\lambda^2 + b\lambda + c)xe^{\lambda x} + (2a\lambda + b)e^{\lambda x} = 0,$ since λ is a root of $a\lambda^2 + b\lambda + c$, and $2a\lambda + b = 2a(-b/2a) + b = 0.$

Therefore, $y_2(x) = xe^{\lambda x}$ is indeed a solution of the differential equation.

The solutions $y_1(x) = e^{\lambda x}$ and $y_2(x) = xe^{\lambda x}$ are linearly independent since they are not proportional.

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The characteristic equation: complex roots

• If $\lambda_{1,2} = \alpha \pm i\beta$ are complex conjugate roots ($\alpha, \beta \in \mathbb{R}$), then two linearly independent solutions are

$y_1^*(x) = e^{(\alpha+i\beta)x}$ and $y_2^*(x) = e^{(\alpha-i\beta)x}$. But they are not real-valued.

Since
$y_1^*(x) = e^{(\alpha+i\beta)x} = e^{\alpha x}e^{i\beta x} = e^{\alpha x}(\cos(\beta x) + i\sin(\beta x))$ and
$y_2^*(x) = e^{(\alpha-i\beta)x} = e^{\alpha x}e^{-i\beta x} = e^{\alpha x}(\cos(\beta x) - i\sin(\beta x))$,
we may reconfigure y_1^*, y_2^* into real-valued functions:

$$y_1 = \frac{1}{2}(y_1^* + y_2^*) = e^{\alpha x}\cos(\beta x),$$

$$y_2 = \frac{1}{2i}(y_1^* - y_2^*) = e^{\alpha x}\sin(\beta x).$$

Finally, two real-valued linearly independent solutions are

$y_1(x) = e^{\alpha x}\cos(\beta x)$ and $y_2(x) = e^{\alpha x}\sin(\beta x)$.

Summary

In order to find the general solution for the differential equation $ay'' + by' + cy = 0$,

1. Compose the *characteristic equation* $a\lambda^2 + b\lambda + c = 0$.

2. Find its roots $\lambda_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

3. Depending on roots, find *linearly independent solutions* y_1, y_2 :

• If λ_1, λ_2 are real and distinct, then $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$.

• If λ is a double root, then $y_1 = e^{\lambda x}$ and $y_2 = xe^{\lambda x}$.

• If $\lambda_{1,2} = \alpha \pm i\beta$ are complex conjugate roots, then $y_1 = e^{\alpha x}\cos(\beta x)$ and $y_2 = e^{\alpha x}\sin(\beta x)$.

4. Compose the *general solution* : $y(x) = C_1y_1 + C_2y_2$, where $C_1, C_2 \in \mathbb{R}$.

Examples: real roots

Example 1. Find the general solution of the equation $y'' + y' - 2y = 0$.

Solution. The characteristic equation is $\lambda^2 + \lambda - 2 = 0 \iff (\lambda + 2)(\lambda - 1) = 0$.
The roots are $\lambda_1 = -2, \quad \lambda_2 = 1$.

The linearly independent solutions are $y_1 = e^{\lambda_1 x} = e^{-2x}, \quad y_2 = e^{\lambda_2 x} = e^x$.

The general solution is $y(x) = C_1 y_1 + C_2 y_2 = C_1 e^{-2x} + C_2 e^x, \quad C_1, C_2 \in \mathbb{R}$.

Example 2. Find the general solution of the equation $y'' - 6y' + 9y = 0$.

Solution. The characteristic equation is $\lambda^2 - 6\lambda + 9 = 0 \iff (\lambda - 3)^2 = 0$.
The double root is $\lambda = 3$.

The linearly independent solutions are $y_1 = e^{\lambda x} = e^{3x}, \quad y_2 = xe^{\lambda x} = xe^{3x}$.

The general solution is $y(x) = C_1 y_1 + C_2 y_2 = C_1 e^{3x} + C_2 x e^{3x}, \quad C_1, C_2 \in \mathbb{R}$.

Examples: complex roots

Example 3. Find the general solution of the equation $y'' + 2y' + 3y = 0$.

Solution. The characteristic equation is

$\lambda^2 + 2\lambda + 3 = 0$.

The roots are $\lambda_{1,2} = \dfrac{-2 \pm \sqrt{4 - 12}}{2} = \dfrac{-2 \pm \sqrt{-8}}{2} = \dfrac{-2 \pm 2i\sqrt{2}}{2} = \underbrace{1}_{\alpha} \pm i \underbrace{\sqrt{2}}_{\beta}$.

The linearly independent solutions are
$y_1 = e^{\alpha x} \cos(\beta x) = e^x \cos(\sqrt{2}x)$ and
$y_2 = e^{\alpha x} \sin(\beta x) = e^x \sin(\sqrt{2}x)$

The general solution is
$y(x) = C_1 y_1 + C_2 y_2 = C_1 e^x \cos(\sqrt{2}x) + C_2 e^x \sin(\sqrt{2}x) = e^x(C_1 \cos(\sqrt{2}x) + C_2 \sin(\sqrt{2}x))$,
$C_1, C_2 \in \mathbb{R}$.