Episode 24

Logistic model

Author of logistic model

Pierre Francois Verhulst (1844)

Areas of applications of logistic model

economics

ecology

demography

sociology

political science

biology

medical science

linguistics

geoscience

Construction of logistic model

A quantity (population size) $y(t)$ changes with time t in such a way that

1. The rate of change $\frac{dy}{dt}$ is proportional to y when y is small:

$$\frac{dy}{dt} \approx ky$$

2. The rate of change $\frac{dy}{dt}$ decreases to 0

when the population size approaches its limit M (*carrying capacity*):

$$\frac{dy}{dt} \underset{y \to M}{\to} 0$$

These two assumptions lead to the *logistic equation:*

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{M}\right)$$

The logistic model is more accurate than the exponential model.

Initial value problem for logistic equation

Given: k (rate of maximum population growth)

M (carrying capacity)

y_0 (initial size of population)

Find: Solution $y = y(t)$ of the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = ky\left(1 - \dfrac{y}{M}\right) \\ y(0) = y_0 \end{cases}$$

Solution. Separate the variables:

$$\frac{dy}{y\left(1 - \dfrac{y}{M}\right)} = k\,dt$$

Perform partial fractions decomposition:

$$\frac{1}{y\left(1 - \dfrac{y}{M}\right)} = \frac{1}{y} + \frac{1}{M-y}$$

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Solving the logistic equation

Integrate:

$$\int \left(\frac{1}{y} + \frac{1}{M-y}\right) dy = \int k\,dt$$

$$\ln|y| - \ln|M-y| = kt + C_1, \quad C_1 \in \mathbb{R}$$

Do some algebra to solve for y:

$$\ln\frac{y}{M-y} = kt + C_1$$

$$\ln\frac{M-y}{y} = -kt - C_1$$

$$\frac{M-y}{y} = Ce^{-kt}, \quad C \in \mathbb{R}$$

$$\frac{M}{y} - 1 = Ce^{-kt}$$

$$y = \frac{M}{1 + Ce^{-kt}}$$

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Solution of the IVP for the logistic equation

The *general solution* of the logistic equation is

$$y = \frac{M}{1 + Ce^{-kt}}, \quad C \in \mathbb{R}$$

To solve the *initial value problem*, let us find C from the initial condition

$$y(0) = y_0$$

$$y_0 = y(0) = \frac{M}{1 + C} \implies C = \frac{M}{y_0} - 1$$

So the solution of the initial value problem for the logistic equation is

$$y(t) = \frac{M}{1 + \left(\frac{M}{y_0} - 1\right) e^{-kt}}$$

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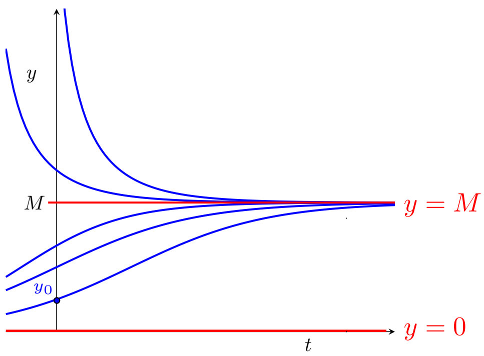
Logistic curves

Solution curves for the logistic equation $\frac{dy}{dt} = ky\left(1 - \frac{y}{M}\right)$



Equilibrium solutions are $y = 0$ and $y = M$.

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Example: Spread of a disease

Problem. A contagious disease is spreading in a town of 10,000 people. There were 200 infected people when the outbreak was discovered, and the number grew up to 1,000 after one month.

Assuming the logistic model for the spread of the disease, find the number of infected people three months after the outbreak. How fast does the disease spread then?

When is the peak of the disease (that is, when does the disease spread most rapidly)? What is the number of infected people then?

Solution.
Let $y(t)$ be the number of infected people t months after the outbreak.

Given: Logistic model $\dfrac{dy}{dt} = ky\left(1 - \dfrac{y}{M}\right)$

Carrying capacity $M = 10,000$

Initial condition $y(0) = 200$

Extra condition $y(1) = 1,000$

What we have to find

Remark: The constant k is unknown,
it will be determined from the given data.

Find:

- The number of infected people after 3 months, $y(3)$
- The rate of spread of the disease after 3 months, $\left.\dfrac{dy}{dt}\right|_{t=3}$
- Time moment t_0 at which $\dfrac{dy}{dt}$ attains its maximum, and $y(t_0)$.

Understanding the logistic model

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{M}\right) \iff \frac{dy}{dt} = \frac{k}{M}\underbrace{y}_{\text{infected}}\underbrace{(M-y)}_{\text{non-infected}}$$

Disclaimer: This is a simplified model for spread of a disease. It doesn't take into account the number of those who are immune
or recovered from the disease.

Solution of IVP

The initial value problem for the logistic equation

$$\begin{cases} \dfrac{dy}{dt} = ky\left(1 - \dfrac{y}{M}\right) \\ y(0) = y_0 \end{cases}$$

has solution

$$y(t) = \frac{M}{1 + \left(\dfrac{M}{y_0} - 1\right)e^{-kt}}$$

In our case,

$$\frac{M}{y_0} = \frac{10,000}{200} = 50$$

So

$$y(t) = \frac{10,000}{1 + 49e^{-kt}}$$

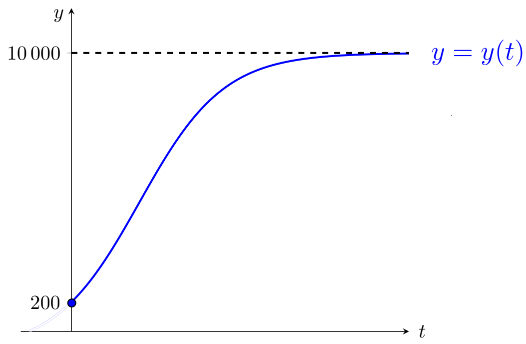
What is k? It will be found from the condition $y(1) = 1,000$:

Solution and solution curve

$$1,000 = y(1) = \frac{10,000}{1 + 49e^{-k}} \implies k = -\ln\frac{9}{49} \approx 1.7$$

Finally, the logistic model for the spread of the disease is

$$y(t) = \frac{10,000}{1 + 49e^{-1.7t}}$$



Answering the questions

Use the logistic solution $y(t) = \dfrac{10,000}{1 + 49e^{-1.7t}}$

to answer the questions about the spread of the disease:

- The number of infected people after 3 months is $y(3) = \dfrac{10,000}{1 + 49e^{-1.7 \cdot 3}} \approx 7,700$ (cases of disease).

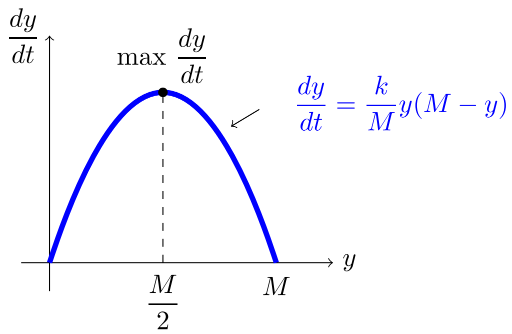
- The rate of spread of the disease at this moment is

$$\left.\frac{dy}{dt}\right|_{t=3} = ky(3)\left(1 - \frac{y(3)}{M}\right) = 1.7 \cdot 7,700\left(1 - \frac{7,000}{10,000}\right) \approx 3,011$$

(cases per month)

Another graphical interpretation of logistic model

• $\max \dfrac{dy}{dt} = ?$ From the logistic equation, $\dfrac{dy}{dt} = \underbrace{ky\left(1 - \dfrac{y}{M}\right)}_{\text{quadratic function in } y}$



$$\max \frac{dy}{dt} = \left.\frac{dy}{dt}\right|_{y=M/2} = \frac{k}{M} \cdot \frac{M}{2}\left(M - \frac{M}{2}\right) = \frac{kM}{4} = \frac{1.7 \cdot 10,000}{4} = 4,250$$

(cases per month)

Maximal rate

The maximal rate of spread of the disease is $4,250$ cases per month.
The number of infected people then is $\dfrac{M}{2} = 5,000$.

When does this peak occur? Find time moment t_0 such that $y(t_0) = 5,000$.

$$5,000 = y(t_0) = \frac{10,000}{1 + 49e^{-1.7t_0}}$$

$$1 + 49e^{-1.7t_0} = 2$$

$$e^{-1.7t_0} = \frac{1}{49}$$

$$t_0 = \frac{2\ln 7}{1.7} \approx 2.3 \text{ (months)}$$

The peak of the disease is expected 2 months and 9 days after the outbreak.

Maximal rate on the logistic curve

Where is the peak of the disease located on the logistic curve?

Peak of the disease occurs when $\dfrac{dy}{dt}$ attains its maximum, that is when $\dfrac{d^2y}{dt^2} = 0$.

On the logistic curve $y = y(t)$, this is an inflection point.