Episode 23: Exponential growth and decay

Biology (population size)
Demography (population size)
Nuclear physics (radioactive decay)
Chemistry (rate of reactions)
Finance (interest rates)

Exponential model: A quantity increases/decreases at a rate proportional to its size.

Let $y(t)$ be the quantity of y at time t.

DE for the model:

$$\frac{dy}{dt} = \underbrace{k}_{\text{a coeff.}} y \qquad \begin{array}{ll} k>0 & \text{growth} \\ k<0 & \text{decay} \end{array}$$

IVP $\begin{cases} \frac{dy}{dt} = ky \\ y(0) = y_0 \end{cases}$

Sol.

$$\frac{dy}{y} = k\,dt$$

$$\ln|y| = kt + C_1$$

$$|y| = e^{kt+C_1}$$

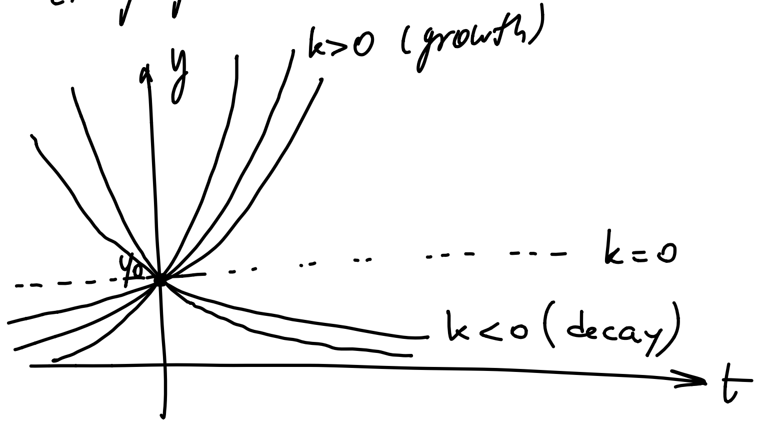
$$y = Ce^{kt}$$

$$y_0 = y(\underset{\uparrow \atop t}{0}) = C\underbrace{e^{k\cdot 0}}_{1} \Rightarrow C = y_0$$

Sol. of the IVP is

$$\boxed{y(t) = y_0 e^{kt}}$$

Exp f-n is the only f-n which rate of growth is proportional to the f-n itself!



Example. After Fukushima nuclear plant explosion (March 12, 2011), cesium-137 has been detected outside the plant. Its half-life is 30 years.

a) Find a formula for the mass left after t years.

b) How much cesium will be left after 100 years out of 200g of initial amount?

c) When will the mass of 200g be reduced to 10g?

Sol.

Model:

let $m(t)$ be the mass ^{137}Cs at time t years after 3.12.2011

$$\text{IVP} \begin{cases} \dfrac{dm}{dt} = km & (k<0) \\ m(0) = m_0 & (\text{initial amount}) \end{cases}$$

$$\frac{dm}{m} = k\,dt$$

$$\ln m = kt + C_1$$

$$m(t) = C e^{kt}$$

$$m_0 = m(\underset{\uparrow\,t}{0}) = C \underbrace{e^{k\cdot 0}}_{1} \Rightarrow C = m_0$$

$$(*)\ \boxed{m(t) = m_0 e^{kt}} \quad k<0$$

a) Half-life of ^{137}Cs is 30 years means that

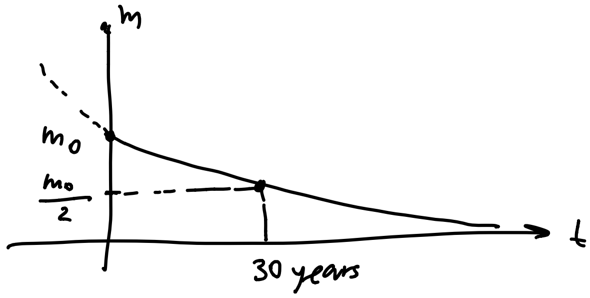
$$m(30) = \frac{m_0}{2} \quad \text{plug into } (*)$$

$$t=30: \quad \underbrace{m(30)}_{\frac{m_0}{2}} = m_0 e^{k\cdot 30}$$

$$\frac{1}{2} = e^{30k}$$

$$k = \frac{\ln(1/2)}{30} = \frac{-\ln 2}{30} \approx -0.023 \quad (<0)$$

$$\boxed{m(t) = m_0 e^{-0.023t}}$$



b) $t = 100$ years

$m_0 = 200g$

$$m(100) = 200 \cdot e^{-0.023\cdot 100} \approx 20\ g$$

c) $t = ?$ s.t. $m(t) = 10\ g$ $m_0 = 200g$

$$\underbrace{m(t)}_{10} = 200\, e^{-0.023t}$$

$$\frac{1}{20} = e^{-0.023t}$$

$$t = \frac{\ln 1/20}{-0.023} \approx 130 \text{ (years)}$$