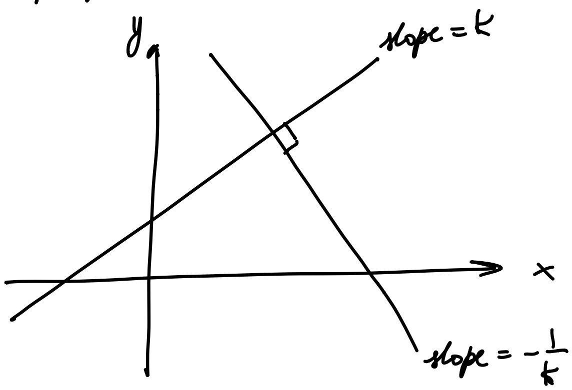
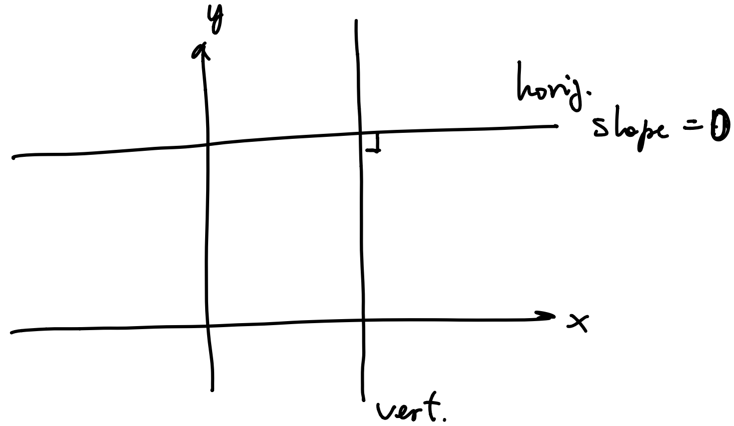
Episode 19. Orthogonal trajectories

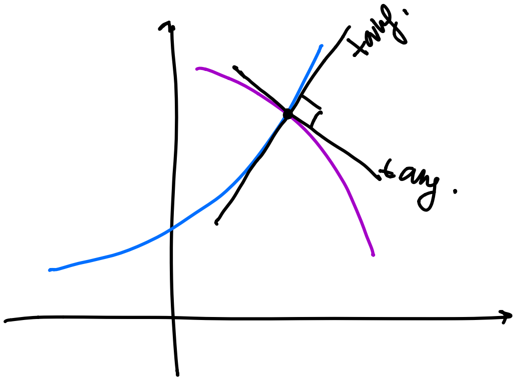
Orthogonal lines
(perpendicular)



or



Orthogonal curves

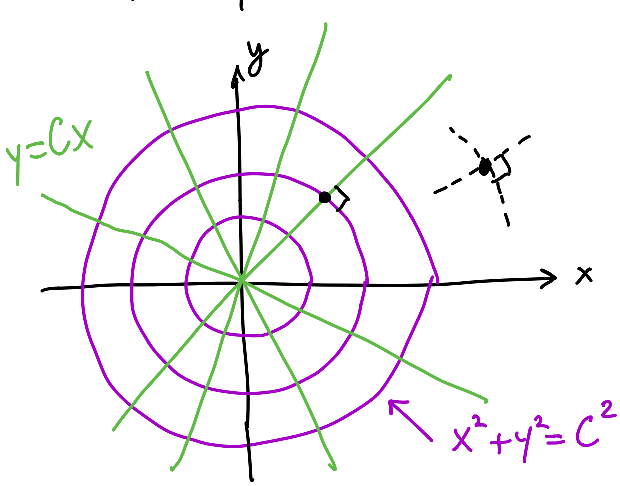


tang. lines at the intersection pt are orthogonal

Ex. Consider two family of curves

1) $x^2+y^2=C^2 \quad (C\in\mathbb{R})$ concentric circles

2) $y=Cx \quad (C\in\mathbb{R})$ lines through origin



At each pt of intersection a circle and a line are orthogonal. Why?

Circles: $x^2+y^2=C^2 \overset{\frac{d}{dx}}{\Rightarrow}$

$2x+2yy'=0 \Rightarrow$

$y'=-\frac{x}{y}$ slope

Lines : $y = Cx \Rightarrow y' = \underbrace{C}_{\frac{y}{x}} \Rightarrow y' = \frac{y}{x}$ slope

the slopes are neg. reciprocal
So the curves are orthog.

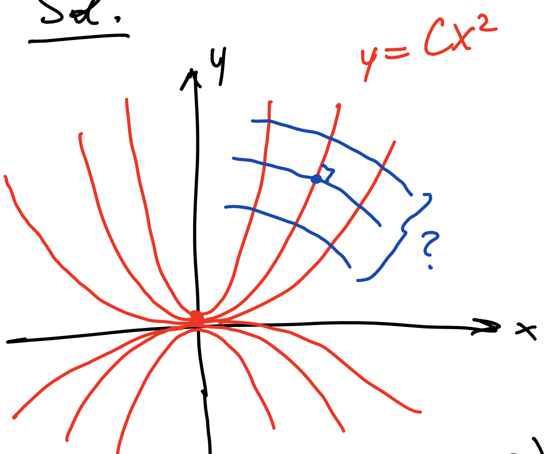
Such families are called
orthogonal trajectories

Problem Given a family of curves.
How to find an orthog. family?

Plan of sol.:
1) Establish a DE for the original family in form $y' = f(x,y)$
2) Compose and solve DE for the orthog. family
$$y' = -\frac{1}{f(x,y)}$$

Example Find the orthog. trajectories to the family of parabolas $y = Cx^2$.

Sol.



1) Isolate C:
$$y = Cx^2 \Rightarrow C = \frac{y}{x^2}$$

2) Differentiate:
$$0 = \frac{y'x^2 - 2xy}{x^4}$$

3) Solve y':
$$y'x^2 - 2xy = 0$$
$$y' = \frac{2xy}{x^2}$$
$$\boxed{y' = \frac{2y}{x}}$$
DE of the original family of parabolas

4) Compose DE for the orthog. family

$$y' = -\frac{x}{2y}$$ ↙ neg. reciprocal

5) Solve this DE

$$y' = -\frac{x}{2y}$$

$$2yy' = -x$$

$$2y\frac{dy}{dx} = -x$$

$$2y\,dy = -x\,dx$$

$$\int 2y\,dy = -\int x\,dx$$

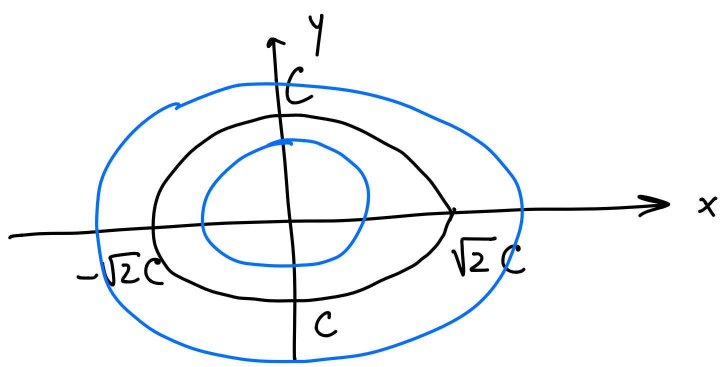
$$y^2 = -\frac{1}{2}x^2 + C_1$$

$$\frac{x^2}{2} + y^2 = C^2, \quad C^2 = C_1$$

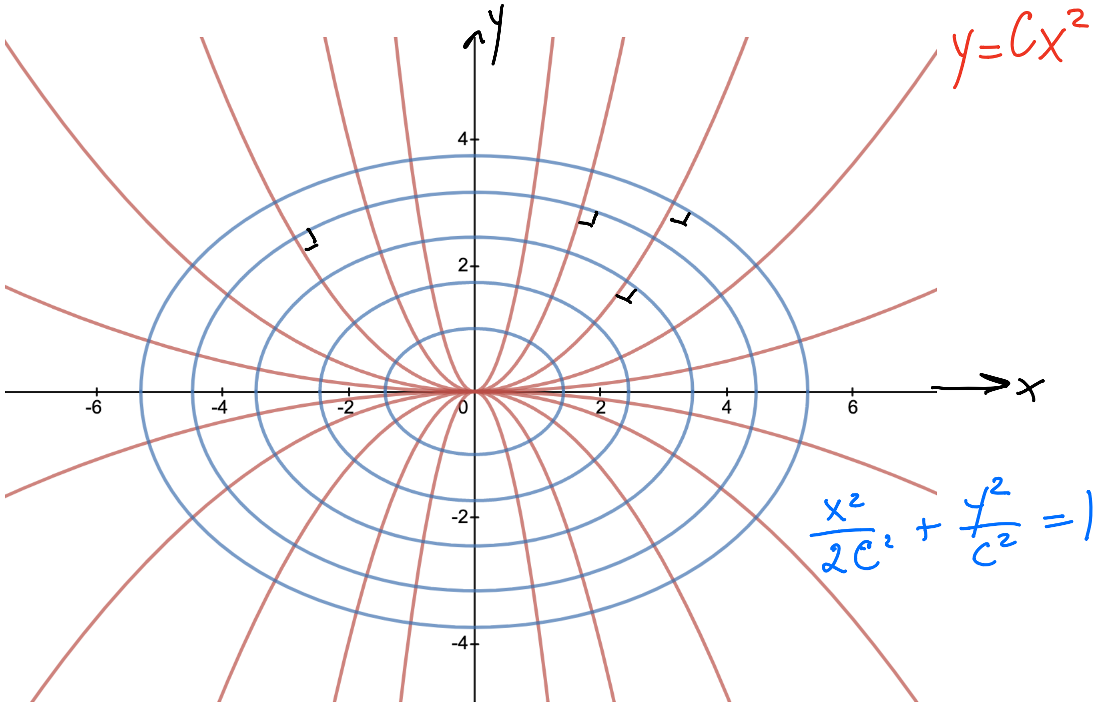
$$\boxed{\frac{x^2}{(\sqrt{2}C)^2} + \frac{y^2}{C^2} = 1}$$

ellipses with semiaxes $\sqrt{2}C$ and C

Answer



Draw both families on the same picture:



$y = Cx^2$
$\frac{x^2}{2C^2} + \frac{y^2}{C^2} = 1$
x
y
-6
-4
-2
0
2
4
6
4
2
-2
-4