Episode 17: Introduction to differential equations. Separable equations

DE

A DE is an equation containing derivative(s) of unknown f-n.

We will study:

- geometric interpretation of DE (direction field, field lines, orthogonal trajectories)
- separable equations
- numerical methods
- applications of DE to special models cooling, mixing, exp. growth/decay, logistic models

<u>Ex.</u> $(*)\ y' = x^2 y$ is a DE. Unknown f-n is $y = y(x)$

Different ways to write $(*)$:

$$y' = x^2 y \iff \frac{dy}{dx} = x^2 y \iff dy = x^2 y\, dx \iff x^2 y\, dx - dy = 0$$

- To solve DE means to find its general solution, that is (all) f-ns satisfying the DE.

How to solve $(*)$

$$y' = x^2 y$$

$$\frac{dy}{dx} = x^2 y$$

$$dy = x^2 y\, dx$$

$$\underbrace{\frac{dy}{y}}_{y \text{ only}} = \underbrace{x^2\, dx}_{x \text{ only}}$$

the variables are <u>separated</u>

Integrate:

$$\int \frac{dy}{y} = \int x^2\, dx$$

$$\ln|y| = \frac{1}{3}x^3 + C_1 \qquad C_1 \in \mathbb{R}$$

$$e^{\ln|y|} = e^{\frac{1}{3}x^3 + C_1}$$

$$|y| = C_2 e^{\frac{1}{3}x^3} \qquad (C_2 = e^{C_1})$$

$$y = \pm C_2 e^{\frac{1}{3}x^3}$$

$$\underbrace{\boxed{y = Ce^{\frac{1}{3}x^3}, \quad C \in \mathbb{R}}}_{\text{a general sol. of } (*)}$$

(C is any real number including 0)

Check the solution:

$$y' = Ce^{\frac{1}{3}x^3} \cdot \left(\frac{1}{3} \cdot 3x^2\right) = Cx^2 e^{\frac{1}{3}x^3}$$

$$y' \stackrel{?}{=} x^2 y$$

$$Cx^2 e^{\frac{1}{3}x^3} = x^2 \cdot Ce^{\frac{1}{3}x^3} \quad \checkmark \quad ☺$$

<u>Initial value problem (IVP) for DE</u>

Find a solution of DE $y' = 3x^2$ satisfying $y(0) = 1$.

$$\text{IVP} \begin{cases} y' = 3x^2 & \text{DE} \\ y(0) = 1 & \text{initial condition} \end{cases}$$

To solve IVP,
1) find a gen. sol. of DE
2) determine the value of the constant

$$y' = 3x^2$$

$$\frac{dy}{dx} = 3x^2$$

$$dy = 3x^2\,dx \quad \text{(the variables are separated)}$$

$$\int dy = \int 3x^2\,dx$$

$$y = x^3 + C \qquad C \in \mathbb{R}$$

a gen. sol.

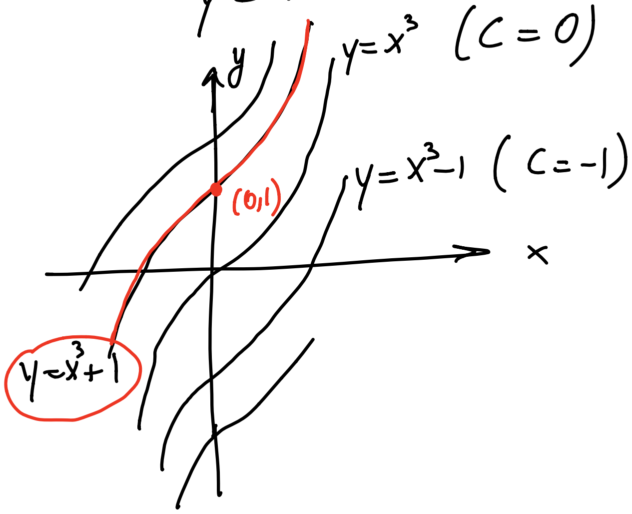
Initial cond: $\underset{\sim}{1} = y(\underset{\uparrow \atop x}{0}) \underset{\uparrow \atop \text{plug in } x=0 \text{ in the gen. sol.}}{=} \underset{\sim}{0^3 + C} \Rightarrow C = 1$

$\boxed{y = x^3 + 1}$ is the sol. of the IVP

Check: $\begin{cases} y' = 3x^2 & \checkmark \\ y(0) = 1 & \checkmark \end{cases}$ ☺

Geometrically, a gen. sol. represents a family of curves (integral curves of DE)

$y = x^3 + C$



IVP $\begin{cases} y' = 3x^2 \\ y(0) = 1 \end{cases}$

To solve the IVP means to find an integral curve passing through $(\underset{\uparrow \atop x}{0}, \underset{\uparrow \atop y}{1})$