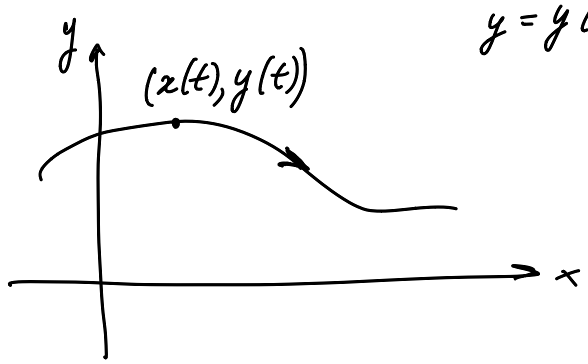
Episode 11: Area enclosed by a parametric curve

$x = x(t)$ $\quad$ $t \in [a,b]$

$y = y(t)$ $\quad$ parameter



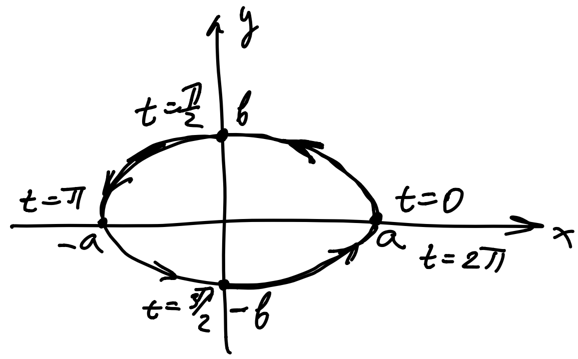
Ex. $\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$ $\quad$ $a, b > 0$ (given const) $\quad$ $t \in [0, 2\pi]$

$\frac{x}{a} = \cos t$

$\frac{y}{b} = \sin t$

$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \cos^2 t + \sin^2 t = 1$

$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$ ellipse



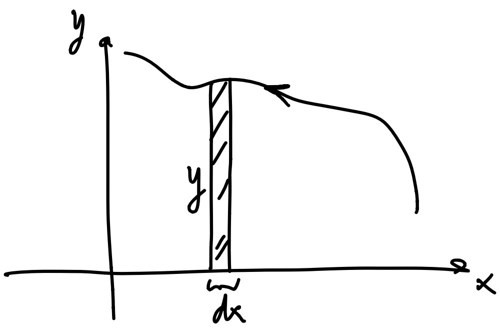
$t = 0$ $\quad$ $x(0) = a\cos 0 = a$

$\qquad\quad$ $y(0) = b\sin 0 = 0$

$t = \frac{\pi}{2}$ $\quad$ $x(\frac{\pi}{2}) = 0$

$\qquad\quad$ $y(\frac{\pi}{2}) = b$

Area = ?

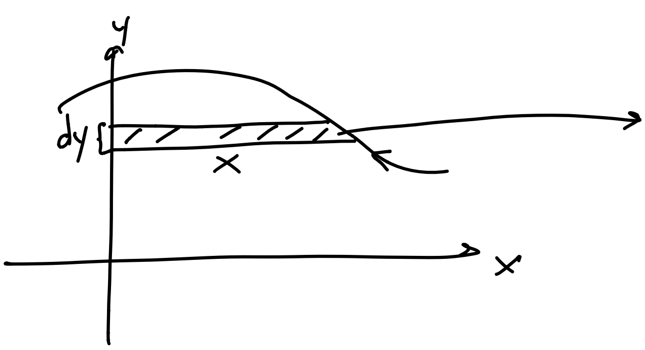


Area element

$dA = \underbrace{y \cdot dx}_{>0}$ if y, dx are both pos. or neg.

or

$dA = -y \cdot dx$ if y, dx are of opposite signs

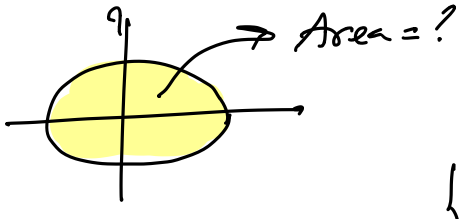


$dA = \underbrace{x\,dy}_{>0}$

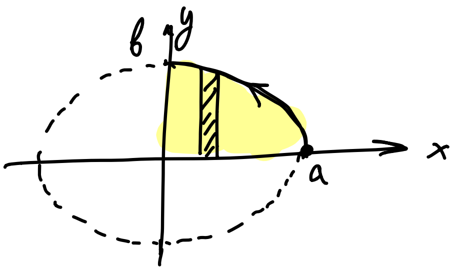
or

$dA = -x\,dy$

Area enclosed by ellipse?



$$\begin{cases} x = a\cos t \\ y = b\sin t \end{cases} \qquad t \in [0, \tfrac{\pi}{2}]$$



Att1

$$dA = -\underbrace{y}_{>0} \cdot \underbrace{dx}_{<0}$$ (since x decreases as t goes from 0 to $\frac{\pi}{2}$)

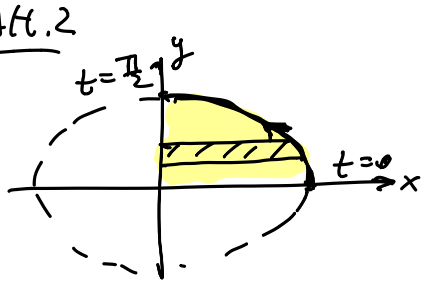
area of 1/4 of elliptic disc

$$A = \int dA = \int_{t=0}^{t=\frac{\pi}{2}} -y\,dx = -\int_0^{\pi/2} \underbrace{b\sin t}_{y} \cdot \underbrace{(-a\sin t)\,dt}_{dx} =$$

$$= ab \int_0^{\pi/2} \sin^2 t \, dt = \qquad \sin^2 t = \frac{1-\cos 2t}{2} \qquad ab \int_0^{\pi/2} \left(\frac{1}{2} - \frac{1}{2}\cos 2t\right) dt =$$

$$ab \left[\frac{1}{2}t - \frac{1}{4}\sin 2t\right]_0^{\pi/2} = ab \, \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi ab}{4}$$ area of 1/4 of ell. disk

Att.2



$$dA = \underbrace{x}_{>0} \cdot \underbrace{dy}_{>0}$$ (y increases when t changes from 0 to $\frac{\pi}{2}$)

$$A = \int x\,dy = \int_0^{\pi/2} \underbrace{a\cos t}_{x} \, \underbrace{(b\cos t)\,dt}_{dy} = \qquad \cos^2 t = \frac{1+\cos 2t}{2}$$

$$ab \int_0^{\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos 2t\right) dt = ab \left[\frac{1}{2}t + \frac{1}{4}\sin 2t\right]_0^{\pi/2} =$$

$$= ab \, \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi ab}{4}$$

Total area enclosed by ellipse $\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$ is

$$\boxed{\pi ab}$$