Episode 10: Area enclosed by a polar curve

a curve in polar coordinates (θ, r)

$r = r(\theta)$ curve in polar coord.

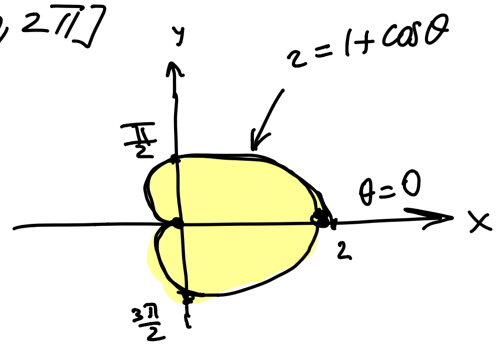
f-n, variable

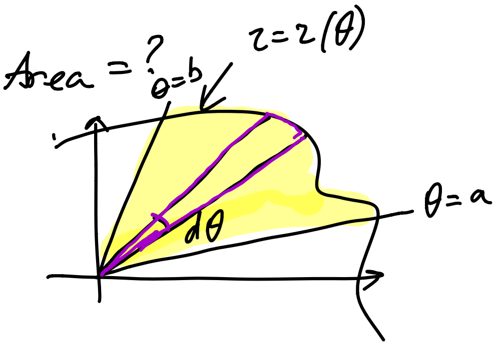
$y = y(x)$ curve in Cart. coord.

Ex. $r = 1 + \cos\theta$, $\theta \in [0, 2\pi]$

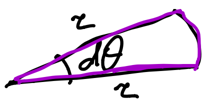
r is a f-n in θ

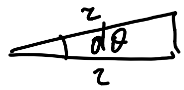
θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r	2	1	0	1	2





area el-t

$dA =$ Area  $\approx$

Area  $= \frac{1}{2} r^2 \underbrace{\sin(d\theta)}_{\approx d\theta \text{ for small } d\theta} \approx$

$= \underbrace{\frac{1}{2} r^2 d\theta}_{\text{area el-t}}$

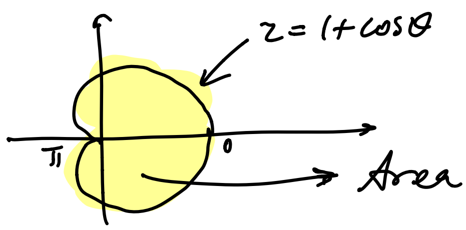
Total area is

$$A = \int dA = \int_{\theta=a}^{\theta=b} \frac{1}{2} r^2 d\theta = \int_{\theta=a}^{\theta=b} \frac{1}{2} \left(r(\theta)\right)^2 d\theta$$

Area in polar coord:

$$\boxed{\int_a^b \frac{1}{2} r^2 d\theta}$$

Find the area enclosed by $\underbrace{r = 1 + \cos\theta}_{\text{cardioid}}$, $\theta \in [0, 2\pi]$



$$\text{Area} = \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2} (1 + \cos\theta)^2 d\theta =$$

$$\underset{\substack{\uparrow \\ \text{symmetry}}}{=} 2\cdot\frac{1}{2}\int_0^{\pi}\left(1+2\cos\theta+\underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}}\right)d\theta = \int_0^{\pi}\left(\frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta\right)d\theta$$

$$= \left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]\Big|_0^{\pi} = \boxed{\frac{3\pi}{2}} \text{ sq. units}$$

area enclosed by the cardioid