# Episode 7: Improper integrals of type I

Improper integrals

Type I

$$\int_a^{\infty}, \quad \int_{-\infty}^{b}, \quad \int_{-\infty}^{\infty}$$

(infinite limits)

Type II

$$\int_a^b f(x)\,dx$$

has an infinite discont. in $[a,b]$

Type I

$$\int_a^{\infty} f(x)\,dx \overset{\text{def.}}{=} \lim_{R\to\infty} \int_a^R f(x)\,dx$$

for $\int_{-\infty}^{b} f(x)dx$, the def. is similar

Ex. 1

$$\int_1^{\infty} \frac{dx}{x^2} = \lim_{R\to\infty} \int_1^R \frac{dx}{x^2} = \lim_{R\to\infty} \left(-\frac{1}{x}\right)\Big|_1^R =$$

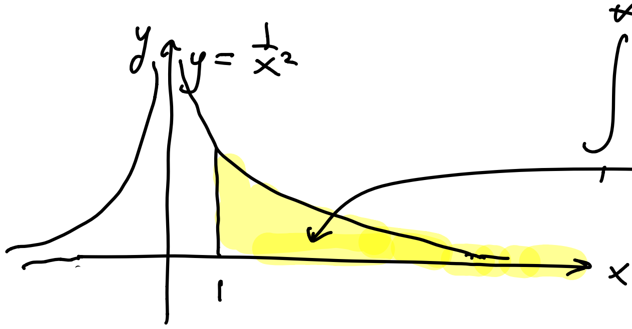
$$= -\lim_{R\to\infty} \left(\frac{1}{R} - 1\right) = -(0-1) = 1$$

A shorter way to write:

$$\int_1^{\infty} \frac{dx}{x^2} = -\frac{1}{x}\Big|_1^{\infty} = -(0-1) = 1$$

$\lim_{x\to\infty} \frac{1}{x}$

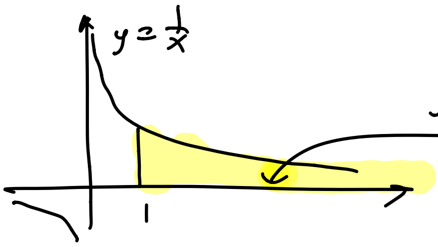
Geom. interpretation:




$$\int_1^\infty \frac{dx}{x^2} = \text{Area}$$

the area of this unbounded region is finite number (1)

Ex. 2 $\displaystyle \int_1^\infty \frac{dx}{x} = \ln|x|\Big|_1^\infty = \infty - 0 = \infty$ the integral diverges (to $\infty$)




$$\int_1^\infty \frac{dx}{x} = \text{Area}$$

this region has infinite area

In general,

p-integral
$$\int_1^\infty \frac{dx}{x^p} \quad \begin{cases} \xrightarrow{p>1} & \text{converges to } \frac{1}{p-1} \\ \xrightarrow{p\le 1} & \text{diverges to } \infty \end{cases}$$

Indeed,

$$\int_1^\infty \frac{dx}{x^p} = \int_1^\infty x^{-p}\,dx = \frac{1}{-p+1}\, x^{-p+1}\Big|_1^\infty =$$

$$= \frac{1}{-p+1}\left[\frac{1}{x^{p-1}}\right]_1^\infty \quad (x>1)$$

$\rightarrow$ if $p>1$ then $\frac{1}{-p+1}[0-1] = \frac{1}{p-1}$

$\rightarrow$ if $p \le 1$ then $\infty$

Ex. 3

$$\int_{x=0}^{x=\infty} \frac{dx}{\sqrt[3]{x+1}} = \begin{bmatrix} u = x+1 \\ du = dx \end{bmatrix} = \int_{u=1}^{u=\infty} \frac{du}{u^{1/3}}$$ diverges to $\infty$ as $p$-integral with $p = \frac{1}{3} \leq 1$

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$$\int_{-\infty}^{\infty} = \int_{-\infty}^{a} + \int_{a}^{\infty}$$ converges if both integral converges

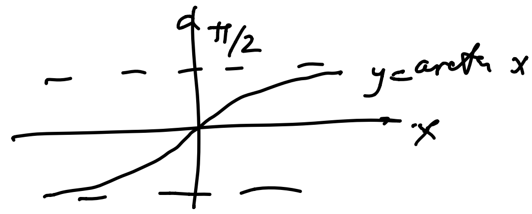
Ex. 4

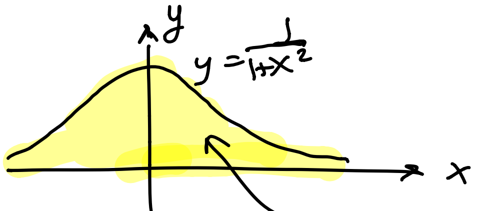
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2} =$$

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \begin{bmatrix} u = -x \\ du = -dx \end{bmatrix} = \int_{u=\infty}^{u=0} \frac{-du}{1+u^2} = -\int_{\infty}^{0} \frac{du}{1+u^2} =$$

$$= \int_{0}^{\infty} \frac{du}{1+u^2} = \int_{0}^{\infty} \frac{dx}{1+x^2}$$

$$= \quad 2 \int_{0}^{\infty} \frac{dx}{1+x^2} = 2 \arctan x \Big|_{0}^{\infty} = 2\left(\frac{\pi}{2} - 0\right) = \pi$$




$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \text{Area} = \pi.$$