Episode 5: Integrating trigonometric functions

① Check symmetry for def. integral

Ex. 1

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x^2 \tan^3 x \, dx = 0$$

$[-\frac{\pi}{4}, \frac{\pi}{4}]$ is $\underset{+}{\underline{\text{symm.}}}$ int. interval

$f(x) = x^2 \tan^3 x$ is an $\underline{\text{odd}}$ f-n $\qquad f(-x) = -f(x)$ for all x

② Use basic trig. identities

$$\left.\begin{aligned} \cos^2 x + \sin^2 x &= 1 \\ \sin x \cos x &= \tfrac{1}{2} \sin 2x \\ \cos^2 x &= \frac{1 + \cos 2x}{2} \\ \sin^2 x &= \frac{1 - \cos 2x}{2} \end{aligned}\right]$$

Ex. 2

$$\int_0^{\pi} \sqrt{1 - \cos x} \, dx = \int_0^{\pi} \sqrt{2 \sin^2\left(\frac{x}{2}\right)} \, dx = \sqrt{2} \int_0^{\pi} \left|\sin \frac{x}{2}\right| dx =$$

$1 - \cos x = 2 \sin^2 \frac{x}{2}$

$x \in [0, \pi]$

$\frac{x}{2} \in [0, \frac{\pi}{2}] \qquad \sin \frac{x}{2} \geq 0$

$$= \sqrt{2} \int_{x=0}^{x=\pi} \sin \frac{x}{2} \, dx = \begin{bmatrix} u = \frac{x}{2} \\ du = \frac{1}{2} dx \end{bmatrix} = \sqrt{2} \int_{u=0}^{u=\frac{\pi}{2}} \sin u \, (2 \, du) =$$

$$= 2\sqrt{2} \left(-\cos u\right)_0^{\frac{\pi}{2}} = -2\sqrt{2} \cos u \Big|_0^{\pi/2} = -2\sqrt{2}\left(\cos \frac{\pi}{2} - \cos 0\right) =$$

$$= -2\sqrt{2}\,(0 - 1) = \underline{2\sqrt{2}}$$

Ex. 3

even power of sin or cos

$$\int \cos^{4} x \, dx = \int \left(\frac{1 + \cos 2x}{2}\right)^2 dx = \frac{1}{4} \int \left(1 + \underline{2\cos 2x} + \cos^2 2x\right) dx =$$

even

$\cos^2 x = \frac{1 + \cos 2x}{2} \qquad \cos^2 2x = \frac{1 + \cos 4x}{2}$

$$= \frac{1}{4} \int \left(1 + 2\cos 2x + \frac{1}{2} + \frac{1}{2} \cos 4x\right) dx =$$

$$\frac{1}{4} \int \left(\frac{3}{2} + 2 \cdot \cos 2x + \frac{1}{2} \cos 4x\right) dx =$$

$= \frac{1}{4} \cdot \left(\frac{3}{2} x + \quad \sin(2x) + \frac{1}{2} \cdot \frac{1}{4} \sin 4x \right) + C =$

$= \frac{3}{8} x + \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x + C$

Ex. 4 $\int \sin^8 x \cos^{3} x \, dx = \int \sin^8 x \underbrace{\cos^2 x}_{1-\sin^2 x} \cdot \cos x \, dx = \begin{bmatrix} u = \sin x \\ du = \cos x \, dx \end{bmatrix} =$

← odd

$\sin^m x \cos^n x$ with m or n odd

$= \int u^8 (1-u^2) \, du = \int u^8 - u^{10} \, du = \frac{1}{9} u^9 - \frac{1}{11} u^{11} + C =$

$= \frac{1}{9} \sin^9 x - \frac{1}{11} \sin^{11} x + C$

Multiplication identities ("product to sum")

✓ $\cos x \cdot \cos y = \frac{1}{2} \left(\cos(x+y) + \cos(x-y) \right)$

$\sin x \cdot \sin y = \frac{1}{2} \left(\cos(x-y) - \cos(x+y) \right)$

$\sin x \cdot \cos y = \frac{1}{2} \left(\sin(x+y) + \sin(x-y) \right)$

Ex. 5 $\int \cos 3x \cdot \cos 5x \, dx = \frac{1}{2} \int \left(\cos 8x + \underbrace{\cos(-2x)}_{\cos 2x} \right) dx =$

$= \frac{1}{2} \left(\frac{\sin 8x}{8} + \frac{\sin 2x}{2} \right) + C =$

$= \frac{1}{16} \sin 8x + \frac{1}{4} \sin 2x + C$

Ex. 6 (all methods together)

$\int \sec x \, dx = \int \frac{dx}{\cos x} = \int \frac{\cos x}{\cos^2 x} \, dx = \int \frac{\cos x \, dx}{1 - \sin^2 x} = \begin{bmatrix} u = \sin x \\ du = \cos x \, dx \end{bmatrix}$

$= \int \frac{du}{1-u^2} =$

$\frac{1}{1-u^2} = \frac{1}{(1-u)(1+u)} = \frac{A}{1-u} + \frac{B}{1+u}$, $A, B = ?$

$$= \frac{A(1+u) + B(1-u)}{(1-u)(1+u)}$$

$$= \frac{(A-B)u + (A+B)}{(1-u)(1+u)}$$

$$\begin{cases} A - B = 0 \\ A + B = 1 \end{cases} \qquad A = B = \frac{1}{2}$$

So $\frac{1}{1-u^2} = \frac{1}{2}\left(\frac{1}{1-u} + \frac{1}{1+u}\right)$

$$= \frac{1}{2}\int \left(\frac{1}{1-u} + \frac{1}{1+u}\right) du = -\frac{1}{2} \ln|1-u| + \frac{1}{2}\ln|1+u| + C$$

$$= \frac{1}{2} \ln\left|\frac{1+u}{1-u}\right| + C = \boxed{\frac{1}{2} \ln\left|\frac{1+\sin x}{1-\sin x}\right|} + C$$

from other sources $\longrightarrow$ $\int \sec x \, dx = \boxed{\ln|\sec x + \tan x|} + C$

$$\frac{1+\sin x}{1-\sin x} = \frac{(1+\sin x)(1+\sin x)}{(1-\sin x)(1+\sin x)} = \frac{(1+\sin x)^2}{1-\sin^2 x} = \frac{(1+\sin x)^2}{\cos^2 x} =$$

$$= \left(\frac{1+\sin x}{\cos x}\right)^2 = (\sec x + \tan x)^2$$

$$\frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| = \frac{1}{2}\ln(\sec x + \tan x)^2 = \ln|\sec x + \tan x|$$