Episode 4: Integrating rational functions

$$\frac{P(x)}{Q(x)} \leftarrow \text{polynomials}$$

- If $\deg P \geq \deg Q$ then divide P by Q

Ex. 1

$$\int \frac{x^3}{x^2+1}\,dx \qquad \begin{matrix} \leftarrow \deg 3 \\ \leftarrow \deg 2 \end{matrix}$$

How to divide
method 1 (short division)

$$\frac{x^3}{x^2+1} = \frac{x(x^2+1)-x}{x^2+1} = x - \frac{x}{x^2+1}$$

method 2 (long division)

$$\begin{array}{r|l} & \ \ x \\ x^2+1 & -\,x^3 \\ & \ \ x^3 \quad +x \\ \hline & \qquad -x \end{array}$$

$$\frac{x^3}{x^2+1} = x - \frac{x}{x^2+1}$$

$$\int \frac{x^3}{x^2+1}\,dx = \int\left(x - \frac{x}{x^2+1}\right)dx = \int x\,dx - \int \frac{x}{x^2+1}\,dx =$$

$$\frac{1}{2}x^2 - \int \frac{du/2}{u} = \underline{\frac{1}{2}x^2 - \frac{1}{2}\ln(x^2+1) + C}$$

$u = x^2+1$
$du = 2x\,dx$

- If $\deg P(x) < Q(x)$

$\deg Q = 1$

Ex. 2

$$\int \frac{2}{3x+4}\,dx = \begin{bmatrix} u = 3x+4 \\ du = 3\,dx \end{bmatrix} = 2\int \frac{du/3}{u} = \frac{1}{6}\ln|3x+4| + C$$

deg 0
deg = 1

$\deg Q = 2$

$$\int \frac{P(x)}{Q(x)} \qquad \begin{matrix} \leftarrow \text{deg 0 or 1} \\ \leftarrow \deg 2 \end{matrix}$$

Roots of Q

2 real root — 1 double root — no real roots

no real roots: completing the square

partial fractions decomposition
(PFD)

- no real roots of Q

EX. 3

$$\int \frac{x-1}{x^2+2x+2}\,dx =$$

$D = 2^2 - 4 \cdot 2 < 0 \Rightarrow x^2+2x+2$ has no real root $\Rightarrow$ do completing the $\square$

$$x^2+2x+2 = \underbrace{(x+1)^2}_{x^2+2x+1} + 1$$

$$= \int \frac{x-1}{(x+1)^2+1}\,dx = \left[\begin{array}{l} u = x+1 \Rightarrow x = u-1 \\ du = dx \end{array}\right]$$

$$= \int \frac{u-1-1}{u^2+1}\,du = \int \frac{u}{u^2+1}\,du - 2\int \frac{du}{u^2+1} =$$

$s = u^2+1$
$ds = 2u\,du$

$$= \int \frac{ds/2}{s} - 2\arctan u = \frac{1}{2}\ln|s| - 2\arctan u + C =$$

$s = u^2+1$

$$= \frac{1}{2}\ln(u^2+1) - 2\arctan u + C =$$

$$= \frac{1}{2}\ln\left((x+1)^2+1\right) - 2\arctan(x+1) + C$$

$u = x+1$

- Q has 2 real roots

EX. 4

$$\int \frac{x+4}{x^2-5x+6}\,dx$$

unknown constants to find

$$\frac{x+4}{x^2-5x+6} = \frac{x+4}{(x-2)(x-3)} \underset{\substack{\uparrow \\ \text{PFD}}}{=} \frac{A}{x-2} + \frac{B}{x-3}$$

How to find A, B:

Method 1: $$\frac{1\cdot x+4}{(x-2)(x-3)} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)}$$

$$= \frac{(A+B)x + (-3A-2B)}{(x-2)(x-3)}$$

coeff. for x^1: coeff for x^0: $\begin{cases} 1 = A+B & \cdot x^2 \, + \\ 4 = -3A-2B \end{cases} \Rightarrow \begin{cases} 6 = -A \\ B = 1-A \end{cases} \quad \begin{cases} A = -6 \\ B = 7 \end{cases}$

solve lin. systems

Method 2:

$$\frac{x+4}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$

mult. by $(x-2)$

$$\frac{x+4}{x-3} = A + \frac{B(x-2)}{x-3}$$

plug in $x=2$:

$$\frac{2+4}{2-3} = A$$

$\underline{A = -6}$

mult. by $(x-3)$

$$\frac{x+4}{x-2} = \frac{A(x-3)}{x-2} + B$$

plug in $x=3$

$$\frac{3+4}{3-2} = B$$

$\underline{B = 7}$

By PFD: $$\frac{x+4}{(x-2)(x-3)} = \frac{-6}{x-2} + \frac{7}{x-3}$$

$$\int \frac{x+4}{(x-2)(x-3)}\,dx = -6\int\frac{dx}{x-2} + 7\int\frac{dx}{x-3} = -6\ln|x-2| + 7\ln|x-3| + C$$

. Q has double root

Ex. 5

$$\int \frac{2x-1}{x^2-6x+9}\,dx$$

$x=3$ is a double root

$$\frac{2x-1}{x^2-6x+9} = \frac{2x-1}{(x-3)^2} \underset{PFD}{=} \frac{A}{x-3} + \frac{B}{(x-3)^2}$$

$A, B = ?$

$$\frac{2x-1}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2}$$

Multi. by $(x-3)^2$:

$$2x-1 = A(x-3) + B$$

Plug in $x=3$

$$2\cdot 3 - 1 = B$$

$$B = 5$$

$$2x-1 = A(x-3) + 5 \qquad A = ?$$

Plug in $x=0$

$$2\cdot 0 - 1 = A(0-3)+5$$

$$-1 = -3A+5$$

$$A = 2$$

PFD: $$\frac{2x-1}{(x-3)^2} = \frac{2}{x-3} + \frac{5}{(x-3)^2}$$

$$\int \frac{2x-1}{(x-3)^2}\,dx = 2\int \frac{dx}{x-3} + 5\int \frac{dx}{(x-3)^2} =$$

$$= 2\ln(x-3) - \frac{5}{x-3} + C$$