Episode 3: Integration by parts

$u = u(x)$
$v = v(x)$

product rule $\quad \frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$

$\downarrow$ integrate

$$uv = \int \frac{du}{dx} v\, dx + \int u \frac{dv}{dx}\, dx$$

$$uv = \int v\, du + \int u\, dv$$

$$\boxed{\int u\, dv = uv - \int v\, du}$$ int. by parts

for def. int.

$$\int_a^b u\, dv = uv\Big|_a^b - \int_a^b v\, du$$

$$\int u\, dv = uv - \int v\, du$$

Ex. 1 $\quad \int x e^x\, dx$

3 ways to write:

1) $\int \underbrace{x}_{u} \underbrace{e^x\, dx}_{dv}$ $\quad \left[\begin{array}{ll} u = x & dv = e^x dx \\ du = dx & v = e^x \end{array} \right] =$

$$\underbrace{x}_{u}\, \underbrace{e^x}_{v} - \int \underbrace{e^x}_{v}\, \underbrace{dx}_{du} = \boxed{xe^x - e^x + C}$$

2) $\displaystyle\int x e^x\,dx = \int \underbrace{x}_{u}\, d(\underbrace{e^x}_{v}) = \underset{u\,v}{x e^x} - \int \underbrace{e^x}_{v}\ \underbrace{dx}_{du} = \underline{x e^x - e^x + C}$

3) $\displaystyle\int x e^x\,dx = x e^x - \int e^x\,dx = \underline{x e^x - e^x + C}$

e^x

1

Ex. 2 (int. by parts twice)

$\sin x$ $\qquad -\cos x$

$$\int x^2 \cos x\,dx = x^2 \sin x - \int \sin x \cdot 2x\,dx = x^2 \sin x - 2\int x \sin x\,dx =$$

$2x$ $\qquad 1$

$$= x^2 \sin x - 2\left[\, x(-\cos x) - \int (-\cos x)\,dx \,\right] =$$

$$\underline{x^2 \sin x + 2x\cos x - 2\sin x + C}$$

Ex. 3 (a trick)

x

$$\int \ln x\,dx = \int \ln x \cdot 1\,dx = x\ln x - \int x \cdot \frac{1}{x}\,dx =$$

$\frac{1}{x}$

$$x\ln x - \int dx = \underline{x\ln x - x + C}$$

Ex. 4 (int. by part + subs)

x

$$\int_0^1 \arctan x\,dx = \int_0^1 \arctan x \cdot 1\,dx = x\arctan x\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx =$$

$\frac{1}{1+x^2}$

$= 1 \cdot \arctan 1 - \frac{1}{2}\ln 2 = \boxed{\frac{\pi}{4} - \frac{1}{2}\ln 2}$

$$\int_{x=0}^{x=1} \frac{x}{1+x^2}\,dx = \left[\begin{array}{l} u = 1+x^2 \\ du = 2x\,dx \Rightarrow x\,dx = du/2 \\ x=0 \Rightarrow u = 1+0^2 = 1 \\ x=1 \Rightarrow u = 1+1^2 = 2 \end{array}\right] = \int_{u=1}^{u=2} \frac{du/2}{u} =$$

$$= \frac{1}{2}\ln|u|\Big|_1^2 = \frac{1}{2}\ln 2$$
