Episode 2: Integration by substitution

$$\int f(u)\,du \underset{\substack{u=g(x)\\ du=g'(x)\,dx}}{=} \int f(g(x))\,g'(x)\,dx$$

Substitution formula:

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du \qquad \text{(u-subs)}$$

for def. int.

$$\int_{x=a}^{x=b} f(g(x))\cdot g'(x)\,dx = \int_{u=g(a)}^{u=g(b)} f(u)\,du$$

When is a subst. used? $u=g(x)$ is obvious

- a new variable
- derivative is present

Ex. 1

$$\int \sin(3x)\,dx = \left[\begin{array}{l} \overset{\text{new}}{u} = \overset{\text{old}}{3x} \\ du = 3\,dx \\ \Rightarrow dx = \frac{du}{3} \end{array}\right] = \int \sin u \,\frac{du}{3} =$$

$$= \frac{1}{3}\int \sin u\,du = -\frac{1}{3}\cos u + C \underset{\substack{\uparrow\\ \text{come back to } x}}{=} \boxed{-\frac{1}{3}\cos(3x) + C}$$

check...

Ex. 2

$$\int_{x=0}^{x=1} \frac{dx}{2x+3} = \left[\begin{array}{l} u = 2x+3 \\ du = 2dx \Rightarrow dx = du/2 \\ x=0 \Rightarrow u = 2\cdot 0+3 = 3 \\ x=1 \Rightarrow u = 2\cdot 1+3 = 5 \end{array}\right] = \int_{u=3}^{u=5} \frac{du/2}{u} =$$

$$= \frac{1}{2}\int_{u=3}^{u=5} \frac{du}{u} = \frac{1}{2}\ln|u|\Big|_3^5 = \frac{1}{2}\left[\ln 5 - \ln 3\right] = \boxed{\frac{\ln 5/3}{2}}$$

Ex. 3

$$\int \frac{\ln^2 x}{x}\,dx = \int \ln^2 x \cdot \underset{\substack{\uparrow\\ \text{driv. of } \ln x}}{\left(\frac{1}{x}\right)} dx = \left[\begin{array}{l} u = \ln x \\ du = \frac{1}{x}\,dx \end{array}\right] =$$

$$= \int u^2\,du = \frac{1}{3}u^3 + C = \boxed{\frac{1}{3}\ln^3 x + C}$$

Ex. 4 (inverse trig. subst.)

$$\int_{x=0}^{x=1} \sqrt{1-x^2}\,dx = \left[\begin{array}{l} \overset{\text{old}}{x} = \overset{\text{new}}{\sin\theta} \\ dx = \cos\theta\,d\theta \\ x=0 \Rightarrow \theta = 0 \\ x=1 \Rightarrow \theta = \frac{\pi}{2} \end{array}\right] = \int_{\theta=0}^{\theta=\frac{\pi}{2}} \underbrace{\cos\theta}_{\sqrt{1-x^2}} \cdot \underbrace{\cos\theta\,d\theta}_{dx} \circledequal$$

idea : $\cos^2\theta + \sin^2\theta = 1$

$\cos^2\theta = 1 - \sin^2\theta$

$$\sqrt{1-x^2} \underset{x=\sin\theta}{=} \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = |\cos\theta| \underset{\theta\in[0,\frac{\pi}{2}]}{=} \cos\theta$$

$$\circledequal \int_0^{\pi/2} \cos^2\theta\,d\theta = \int_0^{\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta =$$

$$\boxed{\cos^2\theta = \frac{1+\cos 2\theta}{2}}$$

$$= \underbrace{\frac{1}{2}\theta\Big|_0^{\pi/2}} + \frac{1}{2}\int_{\theta=0}^{\theta=\pi/2} \cos 2\theta\,d\theta = \frac{\pi}{4} + \frac{1}{2}\int_{u=0}^{u=\pi} \cos u\,\frac{du}{2} =$$

$$\left[\begin{array}{l} 2\theta = u \\ 2\,d\theta = du \end{array}\right]$$

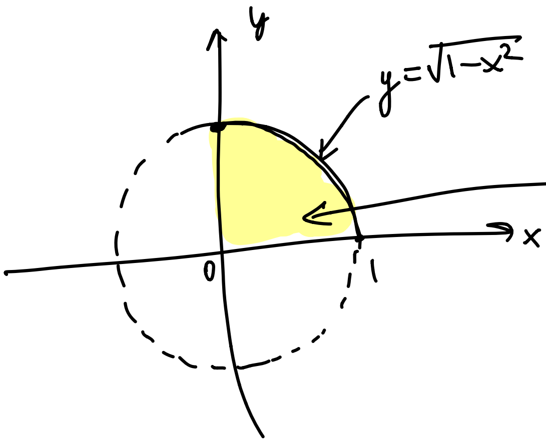
$$= \frac{\pi}{4} + \frac{1}{4}\int_0^{\pi} \cos u\,du = \frac{\pi}{4} + \frac{1}{4}\sin u\Big|_0^{\pi} = \boxed{\frac{\pi}{4}}$$

Geom. interpretation of $\displaystyle\int_0^1 \underbrace{\sqrt{1-x^2}}_{y}\,dx$

$y = \sqrt{1-x^2} \geq 0$

$y^2 = 1 - x^2$

$\boxed{x^2 + y^2 = 1}$ $\quad 0 \leq x \leq 1$



$$\int_0^1 \sqrt{1-x^2}\,dx = \text{Area} = \boxed{\frac{\pi}{4}}$$