MAT131 Homework 33

Problems

1. Evaluae the following indefinite integrals:

   (a) $\int \sin(\pi x)\, dx$

   (b) $\int \frac{4x}{25+9x^2}\, dx$

2. Evaluate the following indefinite integrals:

   (a) $\int \frac{4}{(5+2x)^3}\, dx$

   (b) $\int 10\left(1-2w^2\right) \sqrt[4]{7-3w+2w^3}\, dw$

3. Evaluate the following definite integrals:

   (a) $\int_0^\pi \sec^2(y)\sqrt{2+\tan(y)}\, dy$

   (b) $\int_1^9 \sqrt{x^5} + \frac{\sin(\sqrt{x})}{\sqrt{x}}\, dx$

4. Evaluate the following indefinite integrals:

   (a) $\int \cos^3(12x)\ dx$

   (b) $\int \cos^4(2t)\ dt$

   (c) $\int \cos^2(z)\sin^3(z)\ dz$

5. Use inverse trig substitution to evaulate the following indefinite integral:

$$\int x^3\sqrt{16-x^2}\, dx$$

Answer Key

1. (a) $\displaystyle\int \sin(\pi x)\,dx = -\frac{1}{\pi}\cos(\pi x) + C$

   (b) $\displaystyle\int \frac{4x}{25+9x^2}\,dx = \frac{2}{9}\ln|25+9x^2| + C$

2. (a) $\displaystyle\int \frac{4}{(5+2x)^3}\,dx = -(5+2x)^{-2} + C$

   (b) $\displaystyle\int 10\left(1-2t^2\right)\sqrt[4]{7-3t+2t^3}\,dt = -\frac{8}{3}(7-3t-2t^3)^{\frac{5}{4}} + C$

3. (a) $\displaystyle\int_0^{\frac{\pi}{4}} \sec^2(y)\sqrt{2+\tan(y)}\,dy = 2\sqrt{3} - \frac{4\sqrt{2}}{3}.$

   (b) $\displaystyle\int_1^9 \sqrt{x^5} + \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx = \frac{4372}{7} + 2\cos(1) - 2\cos(3)$

4. (a) $\displaystyle\int \cos^3(12x)\,dx = \frac{1}{12}\sin(12x) - \frac{1}{36}\sin^3(12x) + C$

   (b) $\displaystyle\int \cos^4(2t)\,dt = \frac{3}{8}t + \frac{1}{8}\sin(4t) + \frac{1}{64}\sin(8t) + C$

   (c) $\displaystyle\int \cos^2(z)\sin^3(z)\;dz = -\frac{1}{3}\cos^3(z) + \frac{1}{5}\cos^5(z) + C.$

5. $\displaystyle\int x^3\sqrt{16-x^2}dx = -\frac{1}{15}(16-x^2)^{\frac{3}{2}}\left(32+3x^2\right) + C$

Solutions

1. (a) Let $u = \pi x$. Then $du = \pi dx$. It follows that

$$\int \sin(\pi x)\, dx = \frac{1}{\pi} \int \sin(u)\, du = -\frac{1}{\pi} \cos(u) + C = -\frac{1}{\pi} \cos(\pi x) + C.$$

   (b) Let $u = 25 + 9x^2$. Then $du = 18x dx$. It follows that

$$\int \frac{4x}{25 + 9x^2}\, dx = \frac{2}{9} \int \frac{1}{u}\, du = \frac{2}{9} \ln|u| + C = \frac{2}{9} \ln|25 + 9x^2| + C.$$

2. (a) Let $u = 5 + 2x$. Then $du = 2dx$. It follows that

$$\int \frac{4}{(5+2x)^3}\, dx = 2 \int \frac{1}{u^3} du = -u^{-2} + C = -(5 + 2x)^{-2} + C.$$

   (b) Let $u = 7 - 3t - 2t^3$. Then $du = -3 - 6t^2 dt = -3(1 - 2t^2)dt$. It follows that

$$\int 10 \left(1 - 2t^2\right) \sqrt[4]{7 - 3t + 2t^3}\, dt = -\frac{10}{3} \int u^{\frac{1}{4}} du = -\frac{8}{3} u^{\frac{5}{4}} + C = -\frac{8}{3}(7 - 3t - 2t^3)^{\frac{5}{4}} + C$$

3. (a) Let $u = 2 + \tan(y)$. Then $du = \sec^2(y)dy$. It follows that

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \sec^2(y) \sqrt{2 + \tan(y)}\, dy &= \int_2^3 u^{\frac{1}{2}} du \\ &= \frac{2}{3} u^{\frac{3}{2}} \Big|_2^3 \\ &= \frac{2}{3} \left((3)^{\frac{3}{2}} - (2)^{\frac{3}{2}} \right) \\ &= 2\sqrt{3} - \frac{4\sqrt{2}}{3}. \end{aligned}$$

   (b) Let $u = \sqrt{x}$. Then $du = \frac{1}{2} x^{-\frac{1}{2}} dx$. It follows that

$$\begin{aligned} \int_1^9 \sqrt{x^5} + \frac{\sin(\sqrt{x})}{\sqrt{x}}\, dx &= \int_1^9 x^{\frac{5}{2}}\, dx + 2 \int_1^3 \sin(u)\, du \\ &= \frac{2}{7} x^{\frac{7}{2}} \Big|_1^9 - 2\cos(u) \Big|_1^3 \\ &= \left(\frac{2}{7} (9)^{\frac{7}{2}} - \frac{2}{7} \right) - (2\cos(3) - 2\cos(1)) \\ &= \frac{4372}{7} + 2\cos(1) - 2\cos(3) \end{aligned}$$

4. (a) Notice that

$$\int \cos^3(12x)\,dx = \int \cos^2(12x)\cos(12x)\,dx = \int (1-\sin^2(12x))\cos(12x)/,dx.$$

Let $u = \sin(12x)$. Then $dx = 12\cos(12x)dx$ and it follows that

$$\begin{aligned}
\int (1-\sin^2(12x))\cos(12x)\,dx. &= \frac{1}{12}\int 1-u^2\,du \\
&= \frac{1}{12}\left(u - \frac{1}{3}u^3\right) + C \\
&= \frac{1}{12}\sin(12x) - \frac{1}{36}\sin^3(12x) + C
\end{aligned}$$

(b) We have

$$\begin{aligned}
\int \cos^4(2t)\,dt &= \int \left(\cos^2(2t)\right)^2\,dt \\
&= \int \left(\frac{1+\cos(4t)}{2}\right)^2\,dt \\
&= \int \frac{1}{4} + \frac{1}{2}\cos(4t) + \frac{1}{4}\cos^2(4t)\,dt \\
&= \int \frac{1}{4} + \frac{1}{2}\cos(4t) + \frac{1+\cos(8t)}{8}\,dt \\
&= \int \frac{3}{8} + \frac{1}{2}\cos(4t) + \frac{1}{8}\cos(8t)\,dt \\
&= \frac{3}{8}t + \frac{1}{8}\sin(4t) + \frac{1}{64}\sin(8t) + C
\end{aligned}$$

(c) We have

$$\begin{aligned}
\int \cos^2(z)\sin^3(z)\,dz &= \int \cos^2(z)\sin^2(z)\sin(z)\,dz \\
&= \int \left(\cos^2(z)(1-\cos^2(z))\right)\sin(z)\,dz \\
&= \int \left(\cos^2(z) - \cos^4(z)\right)\sin(z)\,dz
\end{aligned}$$

Let $u = \cos(z)$. Then $dz = -\sin(z)dz$. It follows that

$$\begin{aligned}
\int \left(\cos^2(z) - \cos^4(z)\right)\sin(z)\,dz &= -\int u^2 - u^4 du \\
&= -\frac{1}{3}u^3 + \frac{1}{5}u^5 + C \\
&= -\frac{1}{3}\cos^3(z) + \frac{1}{5}\cos^5(z) + C.
\end{aligned}$$

5. Let $x = 4\sin(\theta)$. Then $dx = 4\cos(\theta)d\theta$. It follows that

$$\begin{aligned}
\int x^3\sqrt{16-x^2}dx &= 256\int \sin^5(\theta)\sqrt{16-16\sin^2(\theta)}\cos(\theta)\,d\theta \\
&= 1024\int \sin^3(\theta)\cos^2(\theta)\,d\theta \\
&= \frac{1024}{5}\cos^5(\theta) - \frac{1024}{3}\cos^3(\theta) + C
\end{aligned}$$

where the last equality holds by Problem $4(c)$.
Since $x = 4\sin(\theta)$,

$$\cos(\theta) = \sqrt{1-\sin^2(\theta)} = \frac{\sqrt{16-x^2}}{4}.$$

It follows that

$$\begin{aligned}
\frac{1024}{5}\cos^5(\theta) - \frac{1024}{3}\cos^3(\theta) + C &= \frac{1}{5}\left(16-x^2\right)^{\frac{5}{2}} - \frac{16}{3}\left(16-x^2\right)^{\frac{3}{2}} + C \\
&= (16-x^2)^{\frac{3}{2}}\left(\frac{16-x^2}{5} - \frac{16}{3}\right) + C \\
&= (16-x^2)^{\frac{3}{2}}\left(\frac{-32-3x^2}{15}\right) + C \\
&= -\frac{1}{15}(16-x^2)^{\frac{3}{2}}\left(32+3x^2\right) + C
\end{aligned}$$