MAT131 Homework 29-30

Problems

1. Given the following table,

x	0	1	2	3	4	5	6
$f(x)$	1	2	5	9	17	26	37

   (a) estimate the area under $f(x)$ between $x = 0$ and $x = 6$ using a left hand Riemann sum with 6 rectangles of equal width,

   (b) estimate the area under $f(x)$ between $x = 0$ and $x = 6$ using a right hand Riemann sum with 6 rectangles of equal width.

   (c) estimate the area under $f(x)$ between $x = 0$ and $x = 6$ using a midpoint Riemann sum with 3 rectangles of equal width.

2. Estimate the area under $f(x) = 4x - x^2$ between $x = 0$ and $x = 4$.

   (a) using n left hand rectangles.

   (b) using n right hand rectangles.

3. Consider the integral $\int_{-9}^{9} \sqrt{81 - x^2}$.

   (a) Express the integral as a limit of Riemann sums.

   (b) Find the exact value of the integral.

4. Use the definition of the integral as a limit of Riemann sums to compute

$$\int_{-2}^{1} 7 - 4x\,dx.$$

5. Compute $\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 - \frac{i^2}{n^2}} \left(\frac{4}{n}\right)$.

Answer Key

1. (a) $L_6 = 60$
   (b) $R_6 = 96$
   (c) $M_3 = 74$
2. (a) $L_n = \dfrac{64n(n-1)}{n^2} - \dfrac{64n(n-1)(2n-1)}{6n^3}$
   (b) $R_n = \dfrac{64n(n+1)}{n^2} - \dfrac{64n(n+1)(2n+1)}{6n^3}$
3. (a) $\displaystyle\int_{-9}^{9} \sqrt{81 - x^2} = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\frac{324i^2}{n^2} - \frac{324i}{n}} \left(\frac{18}{n}\right)$
   (b) $\displaystyle\int_{-9}^{9} \sqrt{81 - x^2} = \frac{81}{2}\pi$
4. $\displaystyle\int_{-2}^{1} 7 - 4x \, dx = 27.$
5. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 - \frac{i^2}{n^2}} \left(\frac{4}{n}\right) = \pi.$

Solutions

1. (a) The width of each rectangle is given by $\Delta x = \frac{6-0}{6} = 1$. The left hand Riemann sum is given by

$$\begin{aligned}(f(0)+f(1)+f(2)+f(3)+f(4)+f(5))\,\Delta x &= (1+2+5+9+17+26)\,(1)\\ &= 60\end{aligned}$$

(b) The width of each rectangle is given by $\Delta x = \frac{6-0}{6} = 1$. The right hand Riemann sum is given by

$$\begin{aligned}(f(1)+f(2)+f(3)+f(4)+f(5)+f(6))\,\Delta x &= (2+5+9+17+26+37)\,(1)\\ &= 96\end{aligned}$$

(c) The width of each rectangle is given by $\Delta x = \frac{6-0}{3} = 2$. The midpoint Riemann sum is given by

$$(f(1)+f(3)+f(5)+)\,\Delta x = (2+9+26)\,(2) \quad = 74$$

2. The width of each rectangle is given by $\Delta x = \dfrac{4-0}{n} = \dfrac{4}{n}$. The i^{th} endpoint of the partition is given by $x_i = \dfrac{4i}{n}$. It follows that:

(a) The left hand Riemann sum is given by

$$\begin{aligned}L_n &= \sum_{i=0}^{n-1} f(x_i)\Delta x\\ &= \sum_{i=0}^{n-1}\left(4\frac{4i}{n} - \left(\frac{4i}{n}\right)^2\right)\frac{4}{n}\\ &= \sum_{i=0}^{n-1}\left(\frac{16i}{n} - \frac{16i^2}{n^2}\right)\frac{4}{n}\\ &= \sum_{i=0}^{n-1}\frac{64i}{n^2} - \frac{64i^2}{n^3}\\ &= \frac{64n(n-1)}{n^2} - \frac{64n(n-1)(2n-1)}{6n^3}\end{aligned}$$

(b) The right hand Riemann sum is given by

$$
\begin{aligned}
L_n &= \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \sum_{i=0}^{n-1} \left(4\frac{4i}{n} - \left(\frac{4i}{n}\right)^2\right)\frac{4}{n} \\
&= \sum_{i=0}^{n-1} \left(\frac{16i}{n} - \frac{16i^2}{n^2}\right)\frac{4}{n} \\
&= \sum_{i=0}^{n-1} \frac{64i}{n^2} - \frac{64i^2}{n^3} \\
&= \frac{64n(n+1)}{n^2} - \frac{64n(n+1)(2n+1)}{6n^3}
\end{aligned}
$$

3. The width of each rectangle is $\Delta x = \frac{9-(-9)}{n} = \frac{18}{n}$. The i^{th} endpoint of the partition is given by $x_i = -9 + \dfrac{18i}{n}$. It follows that the right hand Riemann sum with n-rectangles is given by

$$
\begin{aligned}
R_n &= \sum_{i=1}^{n} \sqrt{81 - \left(-9 + \frac{18i}{n}\right)^2}\left(\frac{18}{n}\right) \\
&= \sum_{i=1}^{n} \sqrt{\frac{324i^2}{n^2} - \frac{324i}{n}}\left(\frac{18}{n}\right).
\end{aligned}
$$

It follows that

$$\int_{-9}^{9} \sqrt{81 - x^2} = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\frac{324i^2}{n^2} - \frac{324i}{n}}\left(\frac{18}{n}\right).$$

4. The definite integral can be interpreted as the area of half of a circle of radius 9. It follows that

$$\int_{-9}^{9} \sqrt{81 - x^2} = \frac{81}{2}\pi.$$

5. We first write down the right hand Riemann sum with n rectangles. The width of each rectangle is given by

$$\Delta x = \frac{1-(-2)}{n} = \frac{3}{n}.$$

The i^{th} endpoint of the partition is given by

$$x_i = -2 + \frac{3i}{n}$$

It follows that the right hand Riemann sum is given by

$$\begin{aligned} R_n &= \sum_{i=1}^{n} f(x_i)\Delta x \\ &= \sum_{i=1}^{n} \left(7 - 4\left(-2 + \frac{3i}{n}\right)\right)\frac{3}{n} \\ &= \sum_{i=1}^{n} \left(15 - \frac{12i}{n}\right)\frac{3}{n} \\ &= \sum_{i=1}^{n} \frac{45}{n} - \frac{36i}{n^2} \\ &= 45 - \frac{36n(n+1)}{2n^2}. \end{aligned}$$

It follows that

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} 45 - \frac{36n(n+1)}{2n^2} = 45 - 18 = 27.$$

6. Notice that

$$\begin{aligned} \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 - \frac{i^2}{n^2}}\left(\frac{4}{n}\right) &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{4 - \frac{4i^2}{n^2}}\left(\frac{2}{n}\right) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{4 - \left(\frac{2i}{n}\right)^2}\left(\frac{2}{n}\right). \end{aligned}$$

We can interpret the limit as a limit of Riemann sums with $\Delta x = \frac{2}{n}$, $x_i = \dfrac{2i}{n}$, and $f(x) = \sqrt{4 - x^2}$. By definition of the definite integral,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{4 - \left(\frac{2i}{n}\right)^2}\left(\frac{2}{n}\right) = \int_0^2 \sqrt{4 - x^2}\, dx.$$

The definite integral on the right can be interpreted as the area of a quarter of a circle of radius 2. It follows that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 - \frac{i^2}{n^2}}\left(\frac{4}{n}\right) = \int_0^2 \sqrt{4 - x^2}\, dx \quad = \frac{1}{4}(4\pi) = \pi.$$