MAT131 Homework 27-28

Problems

1. Evaluate $\int_3^6 |2x - 10| \, dx$ by interpreting the definite integral as a signed area.

2. Suppose

$$f(x) = \begin{cases} \sqrt{25 - x^2} & \text{if } 0 \leq x < 5 \\ x - 5 & \text{if } x \geq 5 \end{cases}$$

   Evaluate $\int_0^7 f(x) \, dx$ by interpreting the definite integral as a signed area.

3. Evaluate the following definite integral:

$$\int_{-1}^{1} \frac{\tan(x)}{x^4 + x^2 + 1} \, dx$$

4. Suppose that $f(x)$ is an even function and $\int_{-10}^{10} f(x) \, dx = 12$. Compute

   (a) $\int_0^{10} f(x) + x \, dx$

   (b) $\int_{-10}^{10} f(x) \sin(x) + x \, dx$

5. If $x + 7 \leq f(x) \leq \sqrt{49 - x^2}$ for all $x \in [-7, 0]$, find upper and lower bounds for $\int_{-7}^{0} f(x) \, dx$.

Answer Key

1. $\displaystyle\int_3^6 |2x-10|\ dx = 5$

2. $\displaystyle\int_0^7 f(x)\,dx = \frac{25}{4}\pi + 2$

3. $\displaystyle\int_{-1}^1 \frac{\tan(x)}{x^4+x^2+1}\,dx = 0$

4. (a) $\displaystyle\int_0^{10} f(x) + x\,dx = 6 + 50 = 56$

   (b) $\displaystyle\int_{-10}^{10} f(x)\sin(x) + x\,dx = 0$

5. $\displaystyle\frac{49}{2} \leq \int_{-7}^0 f(x)\,dx \leq \frac{49}{4}\pi$

Solutions

1. Notice that the integrand can be expressed as

$$|2x-10| = \begin{cases} 10-2x & \text{if } x<5 \\ 2x-10 & \text{if } x \geq 5 \end{cases}$$

   It follows that

$$\int_3^6 |2x-10|\; dx = \int_3^5 10-2x\,dx + \int_5^6 2x-10\,dx$$

   The first integral can be interpreted as the area of a right triangle with height 4 and width 2. It follows that

$$\int_3^5 10-2x\,dx = \frac{1}{2}(4)(2) = 4.$$

   The second integral can be interpreted as the negative of the area of a right triangle with height 2 and width 1. It follows that

$$\int_5^6 2x-10\,dx = \frac{1}{2}(2)(1) = 1.$$

   We conclude that

$$\int_3^6 |2x-10|\; dx = 4+1 = 5.$$

2. We have

$$\int_0^7 f(x)\,dx = \int_0^5 \sqrt{25-x^2}\,dx + \int_5^7 x-5\,dx.$$

   The first integral can be interpreted as the area of a quarter of a circle of radius 5. It follows that

$$\int_0^5 \sqrt{25-x^2}\,dx = \frac{1}{4}(\pi(5)^2) = \frac{25}{4}\pi.$$

   The second integral can be interpreted as the area of a right triangle with height 2 and width 2. It follows that

$$\int_5^7 x-5\,dx = \frac{1}{2}(2)(2) = 2.$$

   We conclude that

$$\int_0^7 f(x)\,dx = \frac{25}{4}\pi + 2.$$

3. Notice that $\dfrac{\tan(x)}{x^4+x^2+1}$ is an odd function. It follows that

$$\int_{-1}^1 \frac{\tan(x)}{x^4+x^2+1}\,dx = 0.$$

4. (a) Notice that

$$\int_0^{10} f(x) + x\,dx = \int_0^{10} f(x)\,dx + \int_0^{10} x\,dx.$$

Since $f(x)$ is an even function,

$$\int_0^{10} f(x)\,dx = \frac{1}{2}\int_{-10}^{10} f(x)\,dx = \frac{1}{2}(12) = 6.$$

The second integral can be interpreted as the area of a right triangle with height 10 and width 10. It follows that

$$\int_0^{10} x\,dx = \frac{1}{2}(10)(10) = 50.$$

It follows that

$$\int_0^{10} f(x) + x\,dx = 6 + 50 = 56.$$

(b) Since $f(x)$ is even and $\sin(x)$ is odd, $f(x)\sin(x)$ is odd. Notice that $g(x) = x$ is also an odd function. It follows that

$$\int_{-10}^{10} f(x)\sin(x) + x\,dx = 0.$$

5. By monotonicity of the integral,

$$\int_{-7}^{0} x + 7\,dx \leq \int_{-7}^{0} f(x)\,dx \leq \int_{-7}^{0} \sqrt{49 - x^2}\,dx$$

The leftmost integral can be interpreted as the area of a right triangle with height 7 and width 7. It follows that

$$\int_{-7}^{0} x + 7 = \frac{1}{2}(7)(7) = \frac{49}{2}.$$

The rightmost integral can be interpreted as the area of a quarter of a circle of radius 7. It follows that

$$\int_{-7}^{0} \sqrt{49 - x^2}\,dx = \frac{1}{4}(\pi(7)^2) = \frac{49}{4}\pi.$$

It follows that

$$\frac{49}{2} \leq \int_{-7}^{0} f(x)\,dx \leq \frac{49}{4}\pi.$$