MAT131 Homework for Lectures 24

July 16, 2021

1 Problems

1. A box with an open top and square base is to be made from 1500 cm^2 of aluminum. Find the dimensions to maxmize the volume and also state what the maximum volume is.

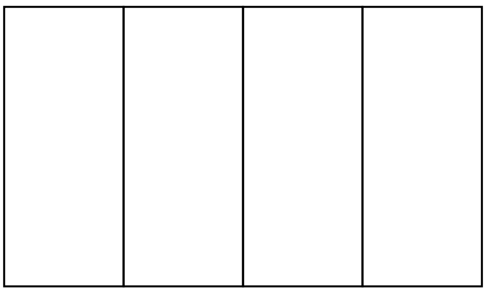
2. A box without a top is to be made from a 12 x 15 inch rectangular sheet of copper. It will be made by cutting out identitcal squares from the corners and then folding the sides up before welding the sides together. What is the maximum volume of the box?

3. A box is to be made in the same way as question 2 but the volume needs to be 130 in^3. What's the least amount of material (in square inches) needed to make this box?

4. A Norman window has two pieces: a semicircle of glass on top of a rectangular pane of glass. The design is such that the perimeter is 10 ft. What should the radius of the semicircular pane be to maximize the area of the window?

5. Consider the collection of rectangles inscribed under the curve $y = \cos(x)$ with base on the x-axis and within the interval $[-\pi/2, +\pi/2]$. Give an equation which determines half the base dimension of the rectangle which maximizes the area within the collection.

6. A corral consists of 4 identical rectangular pens that share fences as in the picture. If 400 m of fence is available, what are the dimensions of the corral and the maximum possible area of the corral?



2 Answer Key

1. $x = \sqrt{500}$ cm, $y = \sqrt{125}$ cm, maximum volume is $500\sqrt{125}$ cm^3.
2. $81 + 21\sqrt{21}$ in^3.
3. $180 - \frac{1}{4}(25 - \sqrt{105})^2$ sq in.
4. $r = 10/(4 + \pi)$ ft.
5. $\cos(x) = x \sin(x)$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
6. Dimensions: 100×40 m; maximum area: 4000 sq m.

3 Solution

1. Let x be the side length and y be the height of the box. Then the volume is given by $V = x^2y$ and the surface area is $x^2 + 4xy$; since we have exactly 1500 cm^2 to use, the constraint equation is $x^2 + 4xy = 1500$.

   Isolate $y = (1500 - x^2)/4x$ and plug this into $V = x^2y = (1500x - x^3)/4$. $\frac{dV}{dx} = (1500 - 3x^2)/4$ tells us how the volume changes as we change x. Setting this equal to zero, we see that the volume is maximized when $x = \sqrt{500}$ cm; the negative square root doesn't make sense in the context of this physical question. Then $y = \sqrt{125}$ cm and the maximum volume is $500\sqrt{125}$ cm^3.

2. There will be 4 squares cut out; say they are of side length x. So the volume of the box would be $V = (12 - 2x)(15 - 2x)x$. So x is also the height of the box. Then $\frac{dV}{dx} = 180 - 108x + 12x^2 = 12(15 - 9x + x^2)$. Setting to zero, the roots are $(9 \pm \sqrt{21})/2$.

   Since the volume is a cubic polynomial with leading term having a positive sign, this means that the smaller root is a local max and the larger root is a local min. Hence, we want $x = (9 - \sqrt{21})/2$ in and the volume will be, after doing arithmetic, $V = 81 + 21\sqrt{21}$ in^3.

3. First, the material needed to make a box with parameter x is $12 * 15 - 4x^2 = 180 - 4x^2$. So the larger the x, the less material needed.

   Now solve $(12 - 2x)(15 - 2x)x - 130 = 0$. Observe that $x = 1$ is a solution so then multiplying out the left hand side to obtain a cubic, we know that it has $(x - 1)$ as a factor. So using polynomial division, we see that the LHS is $(x-1)(2x^2-25x+65)$ and we can find the roots of the quadratic by the quadratic formula. They are $x = (25\pm\sqrt{105})/4$. Note that $\sqrt{105} > 10$ so the positive root is bigger than $(25 + 10)/4 = 35/4 > 8$. Of course, we cannot cut 4 squares with side length bigger than 8 out of the sheet because one side is only 12 inches long and this attempt would require us to cut more than 16 inches off the 12 inch side.

   So the x to minimize material but still create a box with volume 130 in^3 is $x = (25 - \sqrt{105})/4 \approx 3.688$. The amount of copper needed is $180 - \frac{1}{4}(25 - \sqrt{105})^2$ sq inches.

4. Let r be the radius and x be the height of the rectangular pane. Then, the rectangular pane has dimensions $2r \times x$. The perimeter is $\pi r + 2r + 2x = 10$ ft and the area of the window is $A = 2xr + \pi r^2/2$. The constraints show that $x = \frac{1}{2}(10 - (2 + \pi)r)$; plugging this into the formula for area, we get that $A = r(10 - (2 + \pi)r) + \pi r^2/2$. Then $dA/dr = 10 - (4 + \pi)r$ (after arithmetic) and setting this to zero, the value of r that maximizes the area is $r = 10/(4 + \pi)$ ft. We know this is a maximum point because the quadratic describing the area has a minus sign in front of the quadratic term; hence, it is concave down at the critical point.

5. If $(x, 0)$ is the point at which the bottom right corner of the rectangle sits, then the area is given by $A = 2x\cos(x)$. Then $dA/dx = 2\cos(x) - 2x\sin(x)$. Setting this equal to zero, we have that x satisfies $\cos(x) = x\sin(x)$ or $\cot x = x$. So the x value we're looking for us a fixed point of $\cot(x)$ on the proper domain. $\cos(x) = x\sin(x)$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is an acceptable equation.

6. Let x be the width of one of the rectangular enclosures and y the height. So the total area of the corral is $A = 4xy$ and the amount of fencing needed is described by $8x + 5y$ and is constrained to $8x + 5y = 400$. So then $y = (400 - 8x)/5$ and $A = 4x(400 - 8x)/5$. Then $dA/dx = (1600 - 64x)/5$. Setting this equal to zero, we find the critical point is at

$x = 25$ m. This is a maximum since the quadratic function $A(x)$ is concave down. And $y = 40$ m. Hence, the dimensions are 100×40 m and the maximum area is 4000 sq m.