MAT131 Homework for Lectures 23

July 16, 2021

1 Problems

1. A 25 ft ladder is leaning against a vertical wall. The bottom starts to slide away from the wall at 3 ft/s. How fast is the top sliding down when the top is 20 ft above the ground?

2. A conical tank has a height of 18 m and a radius of 6 m; it's positioned so that the nose of the cone is pointing groundward. It is filling with water at a rate of 14π m^3/min. How fast is the height of the water rising when the water is 10 m high?

3. A rocket is launched vertically at 4 mi/s and you're standing 9 mi from the launch site. How fast is the angle of elevation changing after 3 seconds have passed?

4. Air is being pumped into a spherical hot air balloon (made from special elastic material) so that the volume increases at a rate of 10,000 cm^3/s. How fast is the radius of the hot air balloon increasing when the diamater is 20 m? Be careful with units.

5. An ice cube is melting. At the moment when its volume is 27 cm^3, its surface area is decreasing at a rate of 5 cm^2 per minute. At what rate is the volume decreasing at this moment?

2 Answer Key

1. $\frac{dy}{dt} = -\frac{9}{4}$ ft/s
2. $\frac{dh}{dt} = \frac{63}{50}$ m/min
3. $\frac{d\theta}{dt} = \frac{36}{225}$ rad/s
4. $\frac{dr}{dt} = \frac{1}{40{,}000\pi}$ cm/s
5. $\frac{dV}{dt} = -\frac{15}{4}\,\text{cm}^3/\text{min}$

3 Solution

1. Let x represent the horizontal distance between the foot of the ladder and the wall. Let y be the distance between the top of the ladder and the ground. Then, $x^2 + y^2 = 25^2$ is one relation between x and y. Differentiating, we get $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$. On the other hand, when $y = 20$ ft, then $x^2 = 625 - 400 = 225 = 15^2$. So $x = 15$ ft. So plugging in, we have $2(15\text{ ft})(3\text{ ft/s}) + 2(20\text{ ft})\frac{dy}{dt} = 0$. Then solve: $\frac{dy}{dt} = -\frac{9}{4}$ ft/s.

2. As the water rises, the radius and height of the cone of water changes but their ratio does not. The height is always $18/6 = 3$ times the radius. The volume of a cone is $V = \frac{1}{3}\pi r^2 h$. Plugging in that $r = h/3$, we have $V = \frac{1}{27}\pi h^3$ and differentiating, $\frac{dV}{dt} = \frac{1}{9}\pi h^2 \frac{dh}{dt}$. We're told that $\frac{dV}{dt} = 14\pi$ and we're asked to find $\frac{dh}{dt}$ when $h = 10$. Solving, we get $\frac{dh}{dt} = \frac{63}{50}$ m/min.

3. Let x represent the height of the rocket. The relationship between the angle of elevation θ and the height of the rocket is given by $\tan\theta = x/9$. So then differentiating, we have $\frac{dx}{dt} = 9\sec^2\theta\,\frac{d\theta}{dt}$. After 3 seconds, the rocket is 12 miles high. So $\tan\theta = 12/9$ and the hypotenuse of the triangle is 15 mi. So then, $\sec\theta = 15/9$ and hence we have the equation $4 = \frac{225}{9}\frac{d\theta}{dt}$. So $\frac{d\theta}{dt} = \frac{36}{225}$ rad/s.

4. The volume of a sphere is $V = \frac{4}{3}\pi r^3$ and $\frac{dV}{dr} = 4\pi r^2$. Also, the chain rule tells us that $\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt}$. Plugging in the numbers and remembering that 1 m = 100 cm and the radius is half the diameter, we have $10,000\,\text{cm}^3/s = 4\pi(10,000\,\text{cm})^2 \cdot \frac{dr}{dt}$. Then $\frac{dr}{dt} = \frac{1}{40{,}000\pi}$ cm/s. Quite slow.

5. Let x be the side length of the cube. So the volume is $V = x^3$, the surface area is $A = 6x^2$. Then $\frac{dV}{dx} = 3x^2$ and $\frac{dA}{dx} = 12x$. Also, by the chain rule, $\frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt} = 3x^2 \cdot \frac{dx}{dt}$ and $\frac{dA}{dt} = \frac{dA}{dx} \cdot \frac{dx}{dt} = 12x \cdot \frac{dx}{dt}$. At this moment in question, $V = 27$ so $x = 3$.

   Plugging in the values, we have that $\frac{dA}{dt} = -5 = 36 \cdot \frac{dx}{dt}$. So $\frac{dx}{dt} = -\frac{5}{36}$ and hence, $\frac{dV}{dt} = -\frac{15}{4}\,\text{cm}^3/\text{min}$.