MAT131 Homework for Lectures 21-22

July 16, 2021

1 Problems

1. Consider the ellipse in the (x, y)-plane defined by the equation $2x^2 + y^2 - 2xy = 1$. Find the slopes of the tangent lines to the ellipse at the points where they cross the x-axis. Do the same for the tangent lines at the points where the ellipse crosses the y-axis.

2. Find $\frac{d^2y}{dx^2}$ of the equation from Problem 1.

3. Consider the equation $y^2 = x^3$. Graph the solutions to this equation (hint: first consider $y = x^3$). Implicitly differentiate and give an equation for the slope of tangent lines. What happens at the origin? Geometrically, what should the slope of the tangent line be?

4. $$\lim_{x \to 0} \frac{\sin^2 x}{x^2}.$$

5. Let $f(x) = (\sin(x)/x)^2$ be the function from the previous problem. Compute
$$\lim_{x \to 0} f'(x).$$

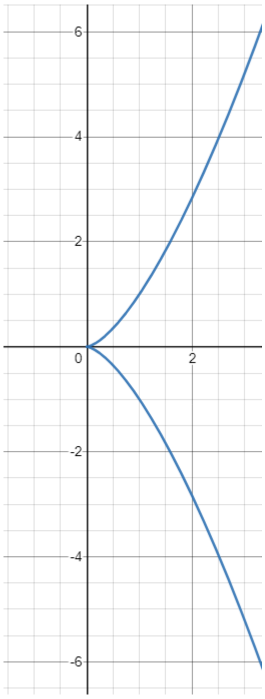
6. $$\lim_{t \to \infty} t \ln(1 + 1/t).$$

2 Answer Key

1. At the x-axis crossings, both have slope 2. At the y-axis crossings, both have slope 1.

2. $y'' = \frac{3y-2x}{(x-y)(y-1)}$

3. $\frac{dy}{dx} = \frac{3x^2}{2y}$. This isn't well-defined at the origin but geometrically, the tangent line should be $y = 0$.



4. 1

5. 0

6. 1

3 Solution

1. The ellipse crosses the x-axis when $y = 0$. So then the equation reduces to $2x^2 = 1$ so the points of crossing are $(\pm 1/\sqrt{2}, 0)$. The ellipses crosses the y-axis at $(0, \pm 1)$.

   Implicitly differentiate the equation:

   $$4x + 2y\frac{dy}{dx} - 2y - 2x\frac{dy}{dx} = 0 \iff \frac{dy}{dx} = \frac{2x - y}{x - y}.$$

   Plugging in $(\pm 1/\sqrt{2}, 0)$, we have that the slope is 2. Plugging in $(0, \pm 1)$, the slope is 1.

2. After the 1st implicit differentiation, we got the equation $4x + 2yy' - 2y - 2yy' = 0$ (for ease of notation, $y' = dy/dx$). Differentiating again:

   $$4 + 2(y')^2 + 2yy'' - 2y' - 2(y')2 - 2y'' = 0.$$

   So then $y'' = \frac{y'-2}{y-1}$. Plugging in what we found for y' before and simplifying,

   $$y'' = \frac{3y - 2x}{(x - y)(y - 1)}.$$

3. For a fixed x_0, if y_0 is a solution to $y_0^2 = x_0^3$, then so is $-y_0$. However, if $x_0 < 0$, then there are no solutions. Hence, the piece of the curve above the x-axis is similar to a cubic graph $z = x^3$ while the piece below is similar to $z = -x^3$; here we have $z = y^2$.

   Implicitly differentiating, we get $\frac{dy}{dx} = \frac{3x^2}{2y}$ which gives an equation for finding the slope of tangents. But it doesn't work when $y = 0$. When $y = 0$, then $x = 0$ as well in order to satisfy $y^2 = x^3$. So at the origin, the slope is not computable from this equation. However, geometrically, a natural tangent line would be $y = 0$.

4. We see that we cannot directly "plug in" 0 but we may use L'Hopital's rule. One application yields $2\sin(x)\cos(x)/2x$. The numerator is equal to $\sin(2x)$ (a trig identity). So the limit is of $\sin(2x)/2x$ which goes to 1 by a second application of L'Hopital's rule or the standard arguments from trigonometry.

5. By the quotient rule,

   $$f'(x) = \frac{x^2\sin(2x) - 2x\sin^2(x)}{x^4} = \frac{x\sin(2x) - 2\sin^2(x)}{x^3}.$$

   Next, we see that we cannot directly plug in $x = 0$ so we apply L'Hopital's rule. Then

   $$\lim_{x\to 0} f'(x) = \lim_{x\to 0} \frac{\sin(2x) + 2x\cos(2x) - 2\sin(2x)}{3x^2}.$$

   We apply L'Hopital's rule again:

   $$\lim_{x\to 0} f'(x) = \lim_{x\to 0} \frac{2\cos(2x) - 4x\sin(2x) - 2\cos(2x)}{6x} = \lim_{x\to 0} \frac{2\sin(2x)}{3} = 0.$$

6. Rewrite the function as $\ln(1+1/t)/(1/t)$. Applying L'Hopital's rule, we get that the limit equals

   $$\lim_{t\to\infty} \frac{\frac{-1/t^2}{1+1/t}}{-1/t^2} = \lim_{t\to\infty} \frac{1}{1 + 1/t} = 1.$$