MAT131 Homework for Lectures 19-20

July 16, 2021

1 Problems

1. Let $f(x) = x^4 - 8x^2 + 10$. Sketch a graph of the function, labeling the local extrema in (x, y)-form.

2. Let $g(x) = \frac{\ln(x)}{x}$. Sketch the graph of the function on its domain of definition, labeling the local extrema in (x, y)-form and also the x-intercept. Also, label the horizontal and vertical asymptotes.

Consider the function $f(x) = e^{1/(x^2+1)}$.

3. Find the 1st derivative of $f(x)$.

4. Find the critical points and values, written in (x, y)-form, of $f(x)$.

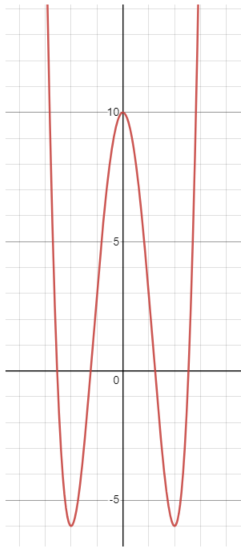
5. Find the 2nd derivative of $f(x)$.

6. Find the critical points of $f'(x)$ and label them as local max/min or inflection point.

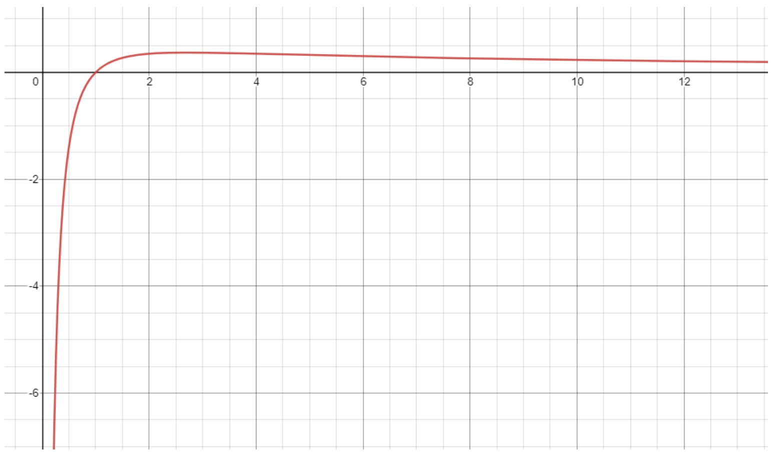
7. Graph $f(x)$ and $f'(x)$ and label critical points and asymptotes.

2 Answer Key

1. Local max: (0,10), absolute min: $(\pm 2, -6)$.



2. Absolute max: $(e, \frac{1}{e})$, x-intercept: $(1, 0)$. Horizontal asymptote: $y = 0$, vertical asymptote: $x = 0$.



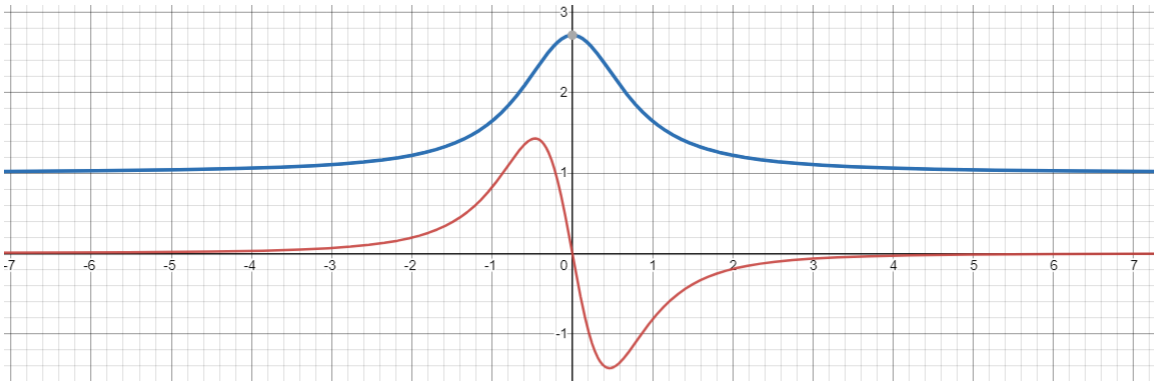
3.
$$f'(x) = -\frac{2x}{(x^2+1)^2}\exp(\frac{1}{x^2+1}).$$

4. $(0, e)$.

5.
$$f''(x) = \left(\frac{6x^4+8x^2-2}{(x^2+1)^4}\right)\exp(\frac{1}{x^2+1}).$$

6. $x = \pm\sqrt{(\sqrt{7}-2)/3}$. The negative root is the local max, the positive is the local min.

7. The blue curve is the graph of $f(x)$, the red is of $f'(x)$.



3
2
1
-1
-7
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7

3 Solution

1. $f'(x) = 4x^3 - 16x = 4x(x^2 - 4)$. So the critical points are $x = \pm 2, 0$. The 2nd derivative is $f''(x) = 12x^2 - 16 = 4(3x^2 - 4)$ and its zeros are $\pm\frac{2}{\sqrt{3}}$. This tells us that the critical points are not inflection points but extrema and concavity tests show that $(0, 10)$ is concave down, hence a local max. The absolute min are $(\pm 2, -6)$. See picture above.

2. $\ln(x)$ is defined on $(0, \infty)$. By quotient rule, $g'(x) = (1 - \ln(x))/x^2$. So its one critical point is at $x = e$. Then $(e, \frac{1}{e})$ is its maximum. This can be checked by observing that $g'(x) > 0$ for $x < e$ and $g'(x) < 0$ for $x > e$.

   $(1, 0)$ is the only x-intercept since $\ln(x) < 0$ for $x < 1$ while $g(x) > 0$ for $x > 1$. $\lim_{x\to\infty} g(x) = 0$ gives $y = 0$ as a horizontal asymptote and $x = 0$ is the vertical asymptote.

3. Write $f(x) = e^{g(x)}$ where $g(x) = (x^2 + 1)^{-1}$; $g'(x) = -2x(x^2 + 1)^{-2}$, and by the quotient rule:

$$g''(x) = \frac{-2(x^2 + 1)^2 - (-2x)2(x^2 + 1)2x}{(x^2 + 1)^4} = \frac{8x^2(x^2 + 1) - 2(x^2 + 1)^2}{(x^2 + 1)^4} = \frac{6x^4 + 4x^2 - 2}{(x^2 + 1)^4}.$$

   Then

$$f'(x) = g'(x)e^{g(x))} = -\frac{2x}{(x^2 + 1)^2} \exp(\frac{1}{x^2 + 1})$$

   by chain rule.

4. The global maximum of $f(x)$ can be found without the 1st derivative simply by noting that $1/(x^2 + 1)$ is largest when $x = 0$ and that e^x is an increasing function. The 1st derivative confirms there is only one critical point and the pair is $(0, e)$.

5. Using chain rule and product rule,

$$f''(x) = g''(x)e^{g(x)} + (g'(x))^2 e^{g(x)} = \left(\frac{6x^4 + 8x^2 - 2}{(x^2 + 1)^4}\right) \exp(\frac{1}{x^2 + 1}).$$

6. Set $f''(x) = 0$. This amounts to solving the quartic equation $3x^4 + 4x^2 - 1 = 0$. However, this reduces to a quadratic by letting $z = x^2$; so we can just use the quadratic formula for $3z^2 + 4z - 1 = 0$. The solutions are

$$z = x^2 = \frac{-2 \pm \sqrt{7}}{3}.$$

   One of these is negative so there are no real solutions $x^2 =$ negative number. But $(\sqrt{7} - 2)/3$ is positive and so this has two real square roots: $\pm\sqrt{(\sqrt{7} - 2)/3}$. Note that plugging in the negative root into $f'(x)$ gives a positive value (just look at the signs) while the positive root gives a negative value.

7. See answer key for graphs.

   The graph of $f(x)$ is straightforward; there is one global max and the function is even and decays rapidly to the value e. So the horizontal asymptote is $y = e$.

For graphing $f'(x)$, note that from above, $3z^2 + 4z - 1$ is quadratic; well, secretly, it is quartic because $z = x^2$ but there are only 2 real roots. So this gives us the rough shape of $f'(x)$: $f'(x)$ is increasing before the negative root and after the positive root but decreasing between the roots. So we can see that $f'(x)$ is an odd function and in fact, the local extrema are also absolute extrema. As $x \to \pm\infty$, $-\frac{2x}{(x^2+1)^2} \to 0$ (bottom power is 4, top power is 1). So the horizontal asymptote for $f'(x)$ is $y = 0$.