MAT131 Homework for Lectures 16-18

July 13, 2021

1 Problems

1. Find the linear approximation of $y = \tan(x)$ at $x = \pi/4$ by giving the equation of a line.
2. Find the linear approximation of the function $f(x) = \sqrt{1 - x^2}$ at $x = 1$ by giving the equation of a line.
3. Find the absolute max/min of $f(x) = 2x^3 - x^2 - 7x - 5$ on the interval $[-2, 3]$.
4. Find the absolute max/min of $f(x) = x^2 e^x$ on the interval $[-1, \infty)$.
5. Use the Mean Value Theorem to show that $f(x) = x^3 - 7x^2 + 25x + 8$ has exactly one real root.

2 Answer Key

1. $y = 1 + 2(x - \pi/4)$
2. $x = 1$
3. Absolute min: $x = (1 + \sqrt{43})/6$, absolute max: $x = 3$
4. Absolute min: $x = 0$, absolute max: $x = -1$
5. See solution

3 Solution

1. The equation of linearization is $L(x) = f(a) + f'(a)(x - a)$.

   $dy/dx = \sec^2(x)$. At $x = \pi/4$, the slope is 2. So the equation of the line is given by $y = 1 + 2(x - \pi/4)$.

2. $f'(x) = -x(1 - x^2)^{-1/2}$. So as $x \to 1$, the slope goes to infinity. This means that the best approximating line is a vertical one: $x = 1$.

3. $f'(x) = 6x^2 - 2x - 7$ and the quadratic formula gives roots $x = (2 \pm \sqrt{4 + 168})/12 = (1 \pm \sqrt{43})/6$. The local max is at the negative root and the local max is at the positive root.

   Note that $6 < \sqrt{43} < 7$ so $-1 < (1 - \sqrt{43})/6 < -5/6 < 0$ and so $f((1 - \sqrt{43})/6)$ is near $f(-1) = -1$. On the otherhand, $f(3) = 19$. So $x = 3$ is the absolute maximum.

   To check the other root, note $\frac{7}{6} < (1 + \sqrt{43})/6 < \frac{8}{6}$. So then, since $(1 + \sqrt{43})/6$ is a local minimum, $f((1 + \sqrt{43})/6)$ must be less than both $f(\frac{7}{6}) = -\frac{613}{54}$ and $f(\frac{8}{6}) = -\frac{307}{27}$. Once can check that both of those values are less than $f(-2) = -11$. Hence, $(1 + \sqrt{43})/6$ is an absolute minimum.

   In conclusion, the local min $x = (1 + \sqrt{43})/6$ is the absolute min but the absolute max is at the endpoint $x = 3$.

4. $f'(x) = 2xe^{-x} - x^2e^{-x} = x(2 - x)e^{-x}$. So the critical points are at $x = 0, 2$. $x = 0$ is a local minimum since clearly $f(x) \geq 0$ and $f(0) = 0$. $f(2) = 4e^{-2}$ where as $f(-1) = e$. Since $2 < e$, $4 < e^2$ and hence $4e^{-2} < 1$. This means $x = -1$ is the global max on $[-1, \infty)$.

5. $f(x)$ is cubic and so $\lim_{x \to \pm\infty} f(x) = \pm\infty$; by the Intermediate Value Theorem, f has at least one real root $x = a$; we'll take a to be the smallest one. Now suppose f has a 2nd real root at $x = b$. So $f(a) = f(b) = 0$.

   By the MVT (or Rolle's theorem), the exists $c \in [a, b]$ such that $f'(c) = \frac{f(b) - f(a)}{b - a} = 0$. That is, c is a critical point of f. Let's compute $f'(x) = 3x^2 - 14x + 25$. Observe that $b^2 - 4ac = 196 - 300 < 0$. So there are no real roots and hence, no critical points. Therefore, there cannot exist such a c.

   An shorter solution is to just directly show that $f(x)$ is always increasing by studying the derivative but the point is to practice using the MVT.