1

Compute the derivative of the following functions:

1. $f(x) = \sin(x^2)\cos(x^2)$

2. $g(x) = \tan(e^{\sin(x)})$

2

Given the function $f(x) = \log_a(x^{\ln(b)})$, prove that $f'(1) = \log_a(b)$.

3

If a function $y = f(x)$ is satisfies $f'(x) > 1$ for all x, is the inverse function $f^{-1}(y)$ increasing or decreasing for all x? Justify your answer.

4

Using logarithmic differentiation, compute the derivative of the function:

$$y = x^x e^x$$

5

If $y = \arctan(x^2 + 3x)$, find the horizontal asymptotes of $\frac{dy}{dx}$. Are there any vertical asymptotes? Justify your answer.

Answer Key

1. (i) $f'(x) = 2x\cos(2x^2)$ (ii) $g'(x) = \frac{e^{\sin(x)}\cos(x)}{\cos^2(e^{\sin(x)})}$.

2. $f'(1) = \frac{\ln(b)}{\ln(a)} = \log_a(b)$.

3. As $0 < (f^{-1})'(y) < 1$, we see that $f^{-1}(y)$ is always increasing.

4. $y' = x^x e^x(\ln(x) + 2)$.

5. Horizontal asymptote at $x = 0$ and no vertical asymptotes.

Solutions

1. Using the double angle identity, $f(x) = \frac{1}{2}\sin(2x^2)$, so that by the chain rule:

$$f'(x) = \frac{1}{2}\cos(2x^2)(4x) = 2x\cos(2x^2)$$

Using the chain rule, we obtain:

$$g'(x) = \sec^2(e^{\sin(x)})(e^{\sin(x)})' = \sec^2(e^{\sin(x)})e^{\sin(x)}\cos(x) = \frac{e^{\sin(x)}\cos(x)}{\cos^2(e^{\sin(x)})}$$

2. We observe that $f(x) = \ln(b)\log_a(x)$, so that:

$$f'(x) = \frac{\ln(b)}{x\ln(a)} \quad \Rightarrow \quad f'(1) = \frac{\ln(b)}{\ln(a)} = \log_a(b)$$

3. Recall that:

$$(f^{-1})'(y) = \frac{1}{f'(x)}$$

whenever $f'(x) \neq 0$, so that if $f'(x) > 1$, we have that $(f^{-1})'(y)$ is well-defined for all $y = f(x)$ and satisfies $0 < (f^{-1})'(y) < 1$. In particular, it is always positive, so $f^{-1}(y)$ is always increasing.

4. Taking the logarithm of both sides gives:

$$\ln(y) = \ln(x^x) + \ln(e^x) = x\ln(x) + x$$

Taking the derivative gives:

$$\frac{y'}{y} = x \cdot \frac{1}{x} + \ln(x) + 1 = \ln(x) + 2$$

Hence:

$$y' = x^x e^x(\ln(x) + 2)$$

5. First, using the chain rule, we compute:

$$\frac{dy}{dx} = \frac{1}{1 + (x^2 + 3x)^2} \cdot (x^2 + 3x)' = \frac{2x + 3}{1 + (x^2 + 3x)^2}$$

To find the horizontal asymptotes, we compute:

$$\lim_{x\to\pm\infty} \frac{dy}{dx} = \lim_{x\to\pm\infty} \frac{2x + 3}{1 + (x^2 + 3x)^2} = \lim_{x\to\pm\infty} \frac{2x}{x^4} = \lim_{x\to\pm\infty} \frac{2}{x^3} = 0$$

Hence, there is only one horizontal asymptote and it is the line $x = 0$. There are no vertical asymptotes, since $(x^2 + 3x)^2 \geq 0$ for all x and so in particular $1 + (x^2 + 3x)^2 \neq 0$ for any x.