1

Compute derivative of the following function:

$$f(x) = e^x(1 + x^2)$$

2

Compute derivative of the following function:

$$g(x) = 2^{x^2} + \frac{x+1}{x^2+1}$$

3

Compute the equation of the tangent line to the curve

$$y = \frac{4x^2+2}{4x^2+3x+2}$$

at the point $x = -1$.

4

Consider the functions:

$$f(x) = \frac{a}{x} \quad \text{and} \quad g(x) = \frac{x+1}{bx}$$

Does the equation $f'(x) = g'(x)$ always have real solutions, for any nonzero values of a and b?

5

Show that the function $y = 3x + \frac{1}{x}$ is a solution to the differential equation:

$$x^2y' - xy + 2 = 0$$

Answer Key

1. $f'(x) = e^x(x^2 + 2x + 1) = e^x(x + 1)^2$.

2. $g'(x) = \ln(2)x2^{x^2+1} + \frac{-x^2-2x+1}{(x^2+1)^2}$.

3. $3y - 2x = 8$.

4. Yes, $x = 0$ is always a solution.

5. Use that $y' = 3 - \frac{1}{x^2}$.

Solutions

1. Using the product rule and power rules, we see that:

$$f'(x) = e^x(1 + x^2)' + (e^x)'(1 + x^2) = 2xe^x + e^x(1 + x^2) = e^x(x^2 + 2x + 1)$$

2. First, using the chain rule, we see that:

$$\frac{d}{dx}(2^{x^2}) = 2^{x^2}\ln(2) \cdot (x^2)' = 2^{x^2}\ln(2) \cdot 2x = \ln(2)x2^{x^2+1}$$

Then, using the sum and quotient rules, we obtain:

$$\begin{aligned} g'(x) &= \ln(2)x2^{x^2+1} + \frac{(x^2+1)(x+1)' - (x+1)(x^2+1)'}{(x^2+1)^2} \\ &= \ln(2)x2^{x^2+1} + \frac{x^2 + 1 - 2x(x+1)}{(x^2+1)^2} \\ &= \ln(2)x2^{x^2+1} + \frac{-x^2 - 2x + 1}{(x^2+1)^2} \end{aligned}$$

3. First, a direct computation shows that $y(-1) = 2$. Using the quotient rule, we compute:

$$\frac{dy}{dx} = \frac{(4x^2+3x+2)(8x) - (4x^2+2)(8x+3)}{(4x^2+3x+2)^2} = \frac{32x^3 + 24x^2 + 16x - 32x^3 - 12x^2 - 16x - 6}{(4x^2+3x+2)^2}$$

so that:

$$\frac{dy}{dx}(-1) = \left.\frac{12x^2 - 6}{(4x^2+3x+2)^2}\right|_{x=-1} = \frac{12-6}{3^2} = \frac{2}{3}$$

Hence, the equation of the tangent line at the point $(-1, 2)$ to the curve $y = y(x)$ is:

$$y - 2 = \frac{2}{3}(x+1) \quad \Rightarrow \quad y = \frac{2}{3}x + \frac{8}{3} \quad \Rightarrow \quad 3y - 2x = 8$$

4. Using the power rule for f and the quotient rule for g, we compute:

$$f'(x) = \frac{-a}{x^2}$$

$$g'(x) = \frac{bx - b(x+1)}{b^2x^2} = \frac{-b}{b^2x^2} = \frac{-1}{bx^2}$$

Hence:

$$f'(x) = g'(x) \quad \Leftrightarrow \quad \frac{-a}{x^2} = \frac{-1}{bx^2} \quad \Leftrightarrow \quad x^2 = abx^2$$

Hence, $x = 0$ is always a solution.

5. We compute:

$$y' = 3 - \frac{1}{x^2}$$

so that:

$$x^2y' - xy + 2 = 3x^2 - 1 - x(3x + \frac{1}{x}) + 2 = (3x^2 - 3x^2) - (1 + 1) + 2 = 2 - 2 = 0$$

verifying that the given curve $y(x)$ is indeed a solution to the given differential equation.