1

Using the definition of the derivative, compute the derivatives of the following functions:

1. $f(x) = 2x + 3$.

2. $g(x) = x^2 + 6x$.

2

Given an example of a function that is continuous everywhere, but not differentiable at the points $x = -1, 0, 1$.

3

Find the equation of the tangent line to the graph of the function $y = x^3 + 1$ at the point $x = 2$.

4

Given an example of a non-linear function whose angle of incline at $x = 0$ is 45 degrees.

5

If a particle's position $p(t)$ in meters at time t in seconds is given by the equation $p(t) = (t + 2)^2 + 4t$, find it's acceleration at time $t = 1$. Is its acceleration constant or nonconstant?

Answer Key

1. (i) $f'(x) = 2$ (ii) $g'(x) = 2x + 6$.

2. Different functions would suffices. One is:

$$f(x) = \begin{cases} |x+1| & x \leq 0 \\ |x-1| & x \geq 0 \end{cases}$$

3. $y = 12x - 15$.

4. Different functions would suffice. One is $f(x) = x^2 + \frac{\sqrt{2}}{2}x$.

5. Acceleration is constantly $2\ ms^{-2}$.

Solutions

1. Using the definition of the derivative, we compute:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{2(x+h) + 3 - (2x+3)}{h} = \lim_{h \to 0} \frac{2h}{h} = \lim_{h \to 0} 2 = 2$$

Using the definition of the derivative, we compute:

$$\begin{aligned} g'(x) &= \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^2 + 6(x+h) - (x^2 + 6x)}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 6x + 6h - x^2 - 6x}{h} \\ &= \lim_{h \to 0} \frac{h(2x + 6 + h)}{h} \\ &= \lim_{h \to 0} (2x + 6 + h) \\ &= 2x + 6 \end{aligned}$$

The technique here was the expand, cancel terms, and then factor out h so that the denominator could be cancelled and the limit could be calculated.

2. We know the absolute value function $|x|$ is everywhere continuous, but is not differentiable at the point $x = 0$. Hence, the shifted absolute value function $|x - 1|$ is not differentiable at the point $x = 1$ and the shifted absolute value function $|x + 1|$ is not differentiable at the point $x = -1$. This inspires the definition of a piecewise function:

$$f(x) = \begin{cases} |x+1| & x \leq 0 \\ |x-1| & x \geq 0 \end{cases}$$

This function is still everywhere continuous (indeed, $|0 + 1| = 1 = |0 - 1|$, so we have continuity at 0 still, the point at which the two absolute value functions are being "glued together"). It is not differentiable at the points $x = \pm 1$. It is also not differentiable at the point $x = 0$, since:

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{|h-1| - 1}{h} = \lim_{h \to 0^+} \frac{|h-1|}{h} = \lim_{h \to 0^+} \frac{1-h}{h} = -1$$

while:

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{|h+1| - 1}{h} = \lim_{h \to 0^-} \frac{|h+1|}{h} = \lim_{h \to 0^+} \frac{h+1}{h} = 1$$

so the limit:

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

does not exist and therefore f is not differentiable at $x = 0$.

3. The derivative is the slope of the tangent line to the curve, so we compute that first:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^3 + 1 - (x^3+1)}{h} \\ &= \lim_{h\to 0} \frac{x^3 + 3hx^2 + 3h^2x + h^3 + 1 - x^3 - 1}{h} \\ &= \lim_{h\to 0} \frac{h(3x^2 + 3hx + h^2)}{h} \\ &= \lim_{h\to 0} (3x^2 + 3hx + h^2) \\ &= 3x^2 \end{aligned}$$

In particular, the slope of the tangent line to the given curve at the point $x = 2$ is $f'(2) = 3(2)^2 = 12$. The equation of the tangent line is then given by:

$$y - f(2) = 12(x-2) \quad \Rightarrow \quad y - 9 = 12(x-2) \quad \Rightarrow \quad y = 12x - 15$$

4. We want a function $f(x)$ such that $f'(0) = \arctan(\pi/4) = \sqrt{2}/2$. Hence, we can take for instance the function $f(x) = x^2 + \frac{\sqrt{2}}{2}x$. Then, a similar computation to $1(ii)$, shows that $f'(0) = 2(0) + \sqrt{2}/2 = \sqrt{2}/2$.

5. Acceleration is the second derivative of position. The first derivative is velocity:

$$\begin{aligned} p'(t) &= \lim_{h\to 0} \frac{p(t+h) - p(x)}{h} \\ &= \lim_{h\to 0} \frac{(t+h+2)^2 + 4(t+h) - ((t+2)^2 + 4t)}{h} \\ &= \lim_{h\to 0} \frac{t^2 + 2(h+2)t + (h+2)^2 + 4t + 4h - (t^2 + 8t + 4)}{h} \\ &= \lim_{h\to 0} \frac{2ht + 8t + h^2 + 8h + 4 - 8t - 4}{h} \\ &= \lim_{h\to 0} \frac{h(2t + 8 + h)}{h} \\ &= \lim_{h\to 0} (2t + 8 + h) \\ &= 2t + 8 \end{aligned}$$

The derivative of $p'(t)$ is $p''(t)$, the acceleration of the particle, and that is then immediately seen to be $p''(t) = 2$. Hence, the acceleration of the particle is constant (the jerk is 0).