1

Calculate the following limits:

1. $\lim_{x\to\infty} \frac{x^2+3x+2}{2x^2+1}$

2. $\lim_{x\to-\infty} e^{x-3}$

2

Calculate the following limit:

$$\lim_{x\to\infty} \frac{\cos(x+2)+x^3}{x^3+1}$$

3

Find the vertical asymptotes of the graph of the function:

$$g(x) = \frac{\tan(x)}{x+1}$$

4

Find all vertical, horizontal, and oblique asymptotes of the graph of the function:

$$h(x) = \frac{3x^4+2x+1}{x^2}$$

5

Find all vertical, horizontal, and oblique asymptotes of the graph of the function:

$$f(x) = \frac{x^5+3x^2+2}{2x^4}$$

Answer Key

1. (i) $1/2$ (ii) 0.

2. 1.

3. $x = 1$ and $x = (2k+1)\pi/2$ for all integers k.

4. Vertical asymptote at $x = 0$; no horizontal asymptotes; no oblique asymptotes.

5. Vertical asymptote at $x = 0$; no horizontal asymptotes; oblique asymptote given by $y = x/2$.

Solutions

1. Only the coefficients of the leading terms matter, so that:

$$\lim_{x\to\infty} \frac{x^2+3x+2}{2x^2+1} = \lim_{x\to\infty} \frac{x^2}{2x^2} = \lim_{x\to\infty} \frac{1}{2} = \frac{1}{2}$$

In the second case, we observe by a graph or by testing values that e^{x-3} is monotonically decreasing as $x \to -\infty$ and bounded below by 0 (indeed, $e^{x-3} > 0$ for all x), so that $\lim_{x\to-\infty} e^{x-3} = 0$.

2. We use the Squeeze Theorem. Since $-1 \leq \cos(x+2) \leq 1$, we have:

$$x^3 - 1 \leq \cos(x+2) + x^3 \leq x^3 + 1$$

so that:

$$\frac{x^3-1}{x^3+1} \leq \frac{\cos(x+2)+x^3}{x^3+1} \leq 1$$

Looking at leading terms, we compute:

$$\lim_{x\to\infty} \frac{x^3-1}{x^3+1} = \lim_{x\to\infty} \frac{x^3}{x^3} = \lim_{x\to\infty} 1 = 1$$

and so by the Squeeze Theorem, we have:

$$\lim_{x\to\infty} \frac{\cos(x+2)+x^3}{x^3+1} = 1$$

3. We must determine the values x' of x for which $\lim_{x\to x'} g(x) = \pm\infty$. First, using the definition of $\tan(x)$, we write:

$$g(x) = \frac{\sin(x)}{\cos(x)(x+1)}$$

This makes it clear that the denominator “blows upäs $x \to -1$, since $-1+1 = 0$ and as $x \to (2k+1)\pi/2$ for any integer k, since $\cos((2k+1)\pi/2) = 0$. Since $\sin(1) \neq 0$ and $\sin((2k+1)\pi/2) = \pm 1$ for all integers k, we see that the limit of $g(x)$ as x approaches any of these values is indeed $\pm\infty$. Away from these points, $g(x)$ is bounded, and so these are the vertical asymptotes of $g(x)$.

4. The vertical asymptote will be at 0, since the numerator evaluates to a finite value while the denominator goes to 0. To check for horizontal asymptotes, we compute:

$$\lim_{x\to\pm\infty} h(x) = \lim_{x\to\pm\infty} \frac{3x^4}{x^2} = \lim_{x\to\pm\infty} 3x^2 = \infty$$

which is not a finite value: hence, there are no horizontal asymptotes. To check for oblique asymptotes to the graph, we compute:

$$\lim_{x\to\infty} \frac{h(x)}{x} = \lim_{x\to\pm\infty} \frac{3x^4}{x^3} = \lim_{x\to\infty} 3x = \infty$$

so again there are no oblique asymptotes (the power in the denominator is too far below the power in the numerator)!

5. The numerator is finite at $x = 0$, but the denominator is 0, so there is again a vertical asymptote at $x = 0$. There is no horizontal asymptote, since:

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x^5}{2x^4} = \lim_{x\to\pm\infty} \frac{x}{2} = \pm\infty$$

There is, however, an oblique asymptote, since:

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{x^5+3x^2+2}{2x^5} = \lim_{x\to\pm\infty} \frac{x^5}{2x^5} = \lim_{x\to\pm\infty} \frac{1}{2} = \frac{1}{2}$$

This is the slope m of the oblique asymptote, and the value b for the line $y = mx + b$ that describes the oblique asymptote is given by:

$$b = \lim_{x\to\infty} (f(x) - mx) = \lim_{x\to\infty} \frac{x^5+3x^2+2-x^5}{2x^4} = \lim_{x\to\infty} \frac{3x^2+2}{2x^4} = \lim_{x\to\infty} \frac{3x^2}{x^4} = \lim_{x\to\infty} \frac{3}{x^2} = 0$$

Thus, the equation of the line of the oblique asymptote is $y = \frac{1}{2}x$.