1

Calculate the following limits:

1. $\lim_{x\to -1} \frac{x^2+2x+1}{2x+2}$

2. $\lim_{x\to -1^+} |x|$

2

Find a value of c that makes the following function continuous:

$$f(x) = \begin{cases} x^2 + c & x \le -2 \\ e^x + cx & x > -2 \end{cases}$$

3

Determine the intervals of continuity of the following function:

$$y = \frac{\cos(x+\pi)}{\sin(x)}$$

4

Compute the following limit:

$$\lim_{x\to 0^+} \frac{\sqrt{x^2+1} - 1}{\sqrt{x}}$$

5

Given that:

$$\lim_{x\to 0} \frac{x^3}{e^x - 1} = 0$$

calculate the following limit:

$$\lim_{x\to 0} \frac{x^3 \cos(\pi x^2)}{e^x - 1}$$

Answer Key

1. (i) 0 (ii) 1.

2. $c = (e^{-2} - 4)/3$.

3. $(\pi k, \pi(k+1))$ for all integers k.

4. 0.

5. 0.

Solutions

1. We cannot compute the limit by directly plugging in values, as we would obtain the indeterminate form $0/0$. However, factoring the denominator, we see that:

$$\lim_{x\to -1} \frac{x^2+2x+1}{2x+2} = \lim_{x\to -1} \frac{(x+1)^2}{2(x+1)} = \lim_{x\to -1} \frac{x+1}{2} = 0$$

The function $f(x) = |x|$ is continuous, so in particular:

$$\lim_{x\to -1^+} |x| = \lim_{x\to -1} |x| = |-1| = 1$$

2. This function is defined piecewise and is continuous whenever $x \neq -2$. To obtain continuity at $x = -2$, we must require that the left and right limits agree, so that:

$$4 + c = (-2)^2 + c = \lim_{x\to -2}(x^2 + c) = \lim_{x\to -2^-} f(x) = \lim_{x\to -2^+} f(x) = \lim_{x\to -2}(e^x + cx) = e^{-2} - 2c$$

Hence, we require a value of c such that:

$$4 + c = e^{-2} - 2c \quad \Rightarrow 3c = e^{-2} - 4$$

So that the value $c = (e^{-2} - 4)/3$ does the trick.

3. The given function is discontinuous whenever $\sin(x) = 0$, which is precisely when $x = \pi k$ for some integer k (at such points, the given function $y = y(x)$ has a vertical asymptote). Hence, the intervals of continuity take the form $(\pi k, \pi(k+1))$ for all integers k.

4. We cannot compute the limit by directly plugging in values, as we would obtain the indeterminate form $0/0$. However, we can rationalize by multiplying by a conjugate:

$$\lim_{x\to 0^+} \frac{\sqrt{x^2+1}-1}{\sqrt{x}} = \lim_{x\to 0^+} \frac{(\sqrt{x^2+1}-1)\cdot(\sqrt{x^2+1}+1)}{\sqrt{x}\cdot(\sqrt{x^2+1}+1)} = \lim_{x\to 0^+} \frac{x^2+1-1}{\sqrt{x}\cdot(\sqrt{x^2+1}+1)} = \lim_{x\to 0^+} \frac{x^{3/2}}{\sqrt{x^2+1}+1} = 0$$

5. The key is to use the Squeeze Theorem. We observe that:

$$-1 \leq \cos(\pi x^2) \leq 1$$

so that:

$$-x^3 \leq x^3 \cos(\pi x^2) \leq x^3$$

and hence:

$$\frac{-x^3}{e^x - 1} \leq x^3 \cos(\pi x^2) \leq \frac{x^3}{e^x - 1}$$

We claim that:

$$\lim_{x\to 0} \frac{-x^3}{e^x - 1} = \lim_{x\to 0} \frac{x^3}{e^x - 1} = 0$$

so that:

$$\lim_{x\to 0} \frac{x^3 \cos(\pi x^2)}{e^x - 1} = 0$$

by the Squeeze Theorem. To prove the claim, we use the hint.