1

Find the coordinates of the vertex and write the equation of the axis of symmetry for the parabola:

$$y = 2x^2 + 4x + 1$$

2

Without using any technological assistants, draw by hand a graph of the function $f(x) = 2\sin(x-1)$, marking the x-intercepts and the maximum and minimum values.

3

Define an inverse function to the function $f : [0, \infty) \to [0, \infty)$ given by $f(x) = x^2$. Does the function $g : [0, \infty) \to \mathbb{R}$ given by $g(x) = x^2$ have an inverse?

4

Let $f(x) = ax + b$, for real numbers a and b, where $a \neq 0, \pm 1$. Find the coordinates of the unique point (x, y) at which the lines $f(x)$ and $f^{-1}(x)$ intersect.

5

Consider the functions $f(x) = \ln(2x-1)$ and $g(x) = \ln(x^2)$. At what values of x do the graphs of these two functions intersect?

Answer Key

1. Vertex: $(-1, 1)$. Axis of Symmetry: $x = -1$.

2. Picture below.

3. $f^{-1}(x) = \sqrt{x}$ while $g(x)$ has no inverse.

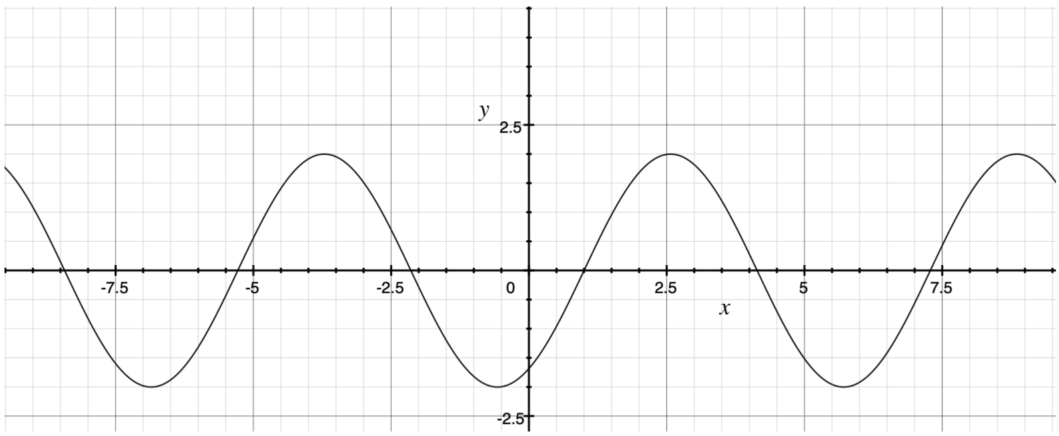
4. $(\frac{b}{1-a}, \frac{b}{1-a})$.

5. $x = 1$.

Solutions

1. The x-coordinate of the vertex is given by $x = -b/2a = -4/4 = -1$. The y-coordinate is $2(-1)^2+4(-1)+1 = 2-4+1 = -1$. Hence, the vertex has coordinates $(-1, 1)$ and the axis of symmetry is the vertical line given by the equation $x = -1$.

2. This is a shifted (by a factor of 1 to the right) and dilated (by a factor of 2) periodic function with period 2π. To draw this, we draw the regular sine function, but now with maximum and miminum values ± 2, respectively, and x-intercepts at $\pi k + 1$ for all integers k.



3. The function f has an inverse $f^{-1} : [0, \infty) \to [0, \infty)$ given by $f^{-1}(x) = \sqrt{x}$, since $(f \circ f^{-1})(x) = (\sqrt{x})^2 = x$ and $(f^{-1} \circ f)(x) = \sqrt{x^2} = x$. However, the function g does not have an inverse, because the square root function is not well-defined on the whole domain $\mathbb{R}$, but g^{-1}, if it existed, would have to be a map $g^{-1} : \mathbb{R} \to [0, \infty)$.

4. The line $f^{-1}(x)$ is given by $f^{-1}(x) = \frac{1}{a}(x - b)$, which is well-defined because $a \neq 0$. A direct computation shows $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$, so f^{-1} is indeed the inverse of f. To find where these two lines intersect, we must solve the equation:

$$ax + b = \frac{1}{a}(x - b)$$

Rearranging, we see that $a^2x + ab = x - b$, so $(a^2 - 1)x = -b(1 + a)$, and therefore we have:

$$x = \frac{-b(1+a)}{a^2 - 1} = \frac{-b(1+a)}{-(1+a)(1-a)} = \frac{b}{1-a}$$

which is well-defined because $a \neq \pm 1$, by hypothesis. Now:

$$f(\frac{b}{1-a}) = \frac{ab}{1-a} + b = \frac{ab + b(1-a)}{1-a} = \frac{b}{1-a}$$

Therefore, the point of intersection is given by:

$$(\frac{b}{1-a}, \frac{b}{1-a})$$

Note that we know a priori by symmetry that this point of intersection (x, y) will satisfy $(x, y) = (y, x)$, so we did not actually have to calculate $f(\frac{b}{1-a})$.

5. We want to find solutions to the equation $\ln(2x - 1) = \ln(x^2)$. Exponentiating, we see that:

$$2x - 1 = e^{\ln(2x-1)} = e^{\ln(x^2)} = x^2 \quad \Rightarrow \quad x^2 - 2x + 1 = 0$$

Factoring gives $x^2 - 2x + 1 = (x - 1)^2 = 0$, so that $f(x) = g(x)$ only when $x = 1$.