1

What is the domain of the function $f(x) = \sqrt{x-1}$? What is the range?

2

Prove that the graph of the function $g(x) = x^3 + x$ is symmetric about the y-axis.

3

Given an example of a non-constant function $h : \mathbb{R} \to \mathbb{R}$ for which $h(x) = h(x+2)$.

4

Consider the function $f(x) = x^3 - x^4$ as a map from $\mathbb{R}$ to $\mathbb{R}$. Show that the range of $f(x)$ is a *proper* subset of the codomain (i.e., not all of $\mathbb{R}$).

5

Prove that the composition of two odd functions is an odd function.

Answer Key

1. $[-1, \infty)$.

2. Suffices to show that g is odd.

3. $h(x) = \sin(\pi x)$.

4. f achieves a global maximum.

5. $f(g(-x)) = f(-g(x)) = -f(g(x))$.

Solutions

1. The function $g(x) = \sqrt{x}$ has domain $[0, \infty)$. In other words, the real-valued square root function is defined for all nonnegative real x values. Hence, the given function $f(x)$ is defined whenever $x - 1 \geq 0$, and so has domain $[-1, \infty)$.

2. It suffices to show that g is an odd function, which is true because:

$$g(-x) = (-x)^3 + (-x) = -x^3 - x = -(x^3 + x) = -g(x)$$

3. Different functions would suffice, but a good option is $h(x) = \sin(\pi x)$. Then, $h(x+2) = \sin(\pi x + 2\pi) = \sin(\pi x)$, since the sine function is 2π-periodic.

4. The key observation is that f achieves a global maximum value. We don't need to calculate this maximum value explicitly to solve this problem. Indeed, we observe that for all $|x| > 1$, we have $x^3 - x^4 < 0$. Moreover, it is certainly the case that when $|x| \leq 1$, we certainly have $f(x) \leq 2$, say. Indeed, $x^3 - x^4 = x^3(1-x)$, and when $|x| \leq 1$, we have $1 - x \leq 2$ while $x^3 \leq 1$, and so $f(x) = x^3(1-x) \leq 2$. Hence, f achieves a global maximum value on the interval $[-1, 1]$ and therefore does not take on every value in the codomain.

5. Let f and g be odd functions. We have:

$$f(g(-x)) = f(-g(x)) = -f(g(x))$$

so that the composition $f \circ g(x) = f(g(x))$ is an odd function. Note that for the first equality we have used the fact that g is odd and for the second we have used the fact that f is odd.