Lecture 33

Integration by Substitution

Objectives

How to evaluate a definite integral?

The most efficient way is to use the Evaluation Theorem:

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b, \text{ where } F(x) \text{ is an antiderivative of } f(x).$$

How do we find an antiderivative? How can we "differentiate backwards"?

$$\underbrace{F(x) = ?}_{\text{to find}} \xrightarrow{\frac{d}{dx}} \underbrace{f(x)}_{\text{given}}$$

There are two major techniques of integration:

integration by substitution and **integration by parts**.

In this lecture, we will discuss the integration by substitution for indefinite and definite integrals.

The substitution rule for the indefinite integral

Suppose that g is a differentiable function, and f is a continuous function.
By the Fundamental Theorem, f has a differentiable antiderivative $F(u)$: $F'(u) = f(u)$.

Consider the composition $F \circ g$. It is differentiable, and, by the **chain rule**,

$$\frac{d}{dx}(F \circ g(x)) = F'(g(x))g'(x) = f(g(x))g'(x), \quad \text{since } F' = f.$$

Let $u = g(x)$, then $\dfrac{du}{dx} = g'(x)$ and $f(g(x))g'(x) = f(u)\dfrac{du}{dx}$.

Let us integrate the last expression:

$$\int f(g(x))g'(x)\,dx = \int f(u)\frac{du}{dx}\,dx \iff \int f(g(x))g'(x)\,dx = \int f(u)du.$$

$$\boxed{\int f(g(x))g'(x)\,dx = \int f(u)du} \quad \text{substitution rule}$$

Substitution in indefinite integrals: examples

1. $\int \sin(2x)\,dx$ Do you see a new variable u? Yes: $u = 2x$.

Recalculate the differential in terms of u:

$u = 2x \implies du = 2dx \implies dx = \frac{1}{2}du$.

Therefore,

$$\int \underbrace{\sin(2x)}_{\sin u} \underbrace{dx}_{\frac{1}{2}du} = \int \frac{1}{2}\sin u\,du = -\frac{1}{2}\cos u + C$$

[now we go back to x to finish the problem] $= -\frac{1}{2}\cos(2x) + C$.

⚠ **Warnings:**

1. Each integral should contain either x or u, but not both!

2. The answer should use **the same** variable as the original integral.

Substitution in indefinite integrals: examples

2. $\int \frac{1}{3x+2}\,dx$ Do you see a new variable u? Yes: $u = 3x + 2$.

$$\int \frac{1}{3x+2}\,dx \left[\begin{array}{l} u = 3x+2 \\ du = 3dx \implies dx = du/3 \\ \frac{1}{3x+2} = \frac{1}{u} \end{array}\right] = \int \frac{1}{u}\frac{du}{3} = \frac{1}{3}\ln|u| + C$$

$$= \frac{1}{3}\ln|3x+2| + C.$$

3. $\int \tan x\,dx$ Do you see a new variable u? Hmm...

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx \left[\begin{array}{l} u = \cos x \\ du = -\sin x dx \\ \frac{\sin x\,dx}{\cos x} = \frac{-du}{u} \end{array}\right] = -\int \frac{du}{u} = -\ln|u| + C$$

$$= -\ln|\cos x| + C = \ln\frac{1}{|\cos x|} + C.$$

The substitution rule for the definite integral

In the substitution formula for the indefinite integral

$$\int f(g(x))g'(x)\,dx = \int f(u)du\,,$$

$u = g(x)$ is a new variable for integration.

In a definite integral, if x runs from a to b,

then u runs from $u = g(a)$ to $u = g(b)$, and we find:

$$\boxed{\int_{x=a}^{x=b} f(g(x))g'(x)\,dx = \int_{u=g(a)}^{u=g(b)} f(u)du} \quad \text{substitution rule}$$

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Substitution in definite integrals: examples

1. $\displaystyle\int_{x=1}^{x=2} \frac{\ln x}{x}\,dx \left[\begin{array}{l} u = \ln x \\ du = \frac{1}{x}dx \\ x = 1 \implies u = \ln 1 = 0 \\ x = 2 \implies u = \ln 2 \end{array}\right] = \int_{u=0}^{u=\ln 2} u\,du = \frac{1}{2}u^2\Big|_0^{\ln 2} = \frac{1}{2}\ln^2 2$

2. $\displaystyle\int_{x=-1}^{x=0} x^5(2x^6+3)^7\,dx \left[\begin{array}{l} u = 2x^6+3 \\ du = 12x^5dx \implies x^5dx = du/12 \\ x = -1 \implies u = 2(-1)^6+3 = 5 \\ x = 0 \implies u = 2\cdot 0^6+3 = 3 \end{array}\right]$

$$= \int_{u=5}^{u=3} u^7\,\frac{du}{12} = \frac{1}{96}u^8\Big|_5^3 = \frac{3^8-5^8}{96}$$

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Substitution in definite integrals: examples

3. $\displaystyle\int_{x=0}^{x=1} x\sqrt{1+x^2}\,dx \quad \begin{bmatrix} u = 1+x^2 & \\ du = 2xdx \implies xdx = \dfrac{du}{2} & \\ x=0 \implies u = 1+0^2 = 1 & \\ x=1 \implies u = 1+1^2 = 2 & \end{bmatrix} = \int_{u=1}^{u=2} \sqrt{u}\,\frac{du}{2}$

$$= \frac{1}{2}\int_{u=1}^{u=2} u^{1/2}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_1^2 = \frac{1}{3}(2^{3/2}-1) = \frac{2\sqrt{2}-1}{3}$$

4. $\displaystyle\int_{x=0}^{x=1/3} \frac{1}{9x^2+1}\,dx = \int_{x=0}^{x=1/3} \frac{1}{(3x)^2+1}\,dx \quad \begin{bmatrix} u = 3x & \\ du = 3dx \implies dx = \dfrac{du}{3} & \\ x=0 \implies u = 3\cdot 0 = 0 & \\ x=1/3 \implies u = 3\cdot\frac{1}{3} = 1 & \end{bmatrix}$

$$= \frac{1}{3}\int_{u=0}^{u=1} \frac{1}{u^2+1}\,du = \frac{1}{3}\arctan u\Big|_{u=0}^{u=1} = \frac{1}{3}(\arctan 1 - \arctan 0) = \frac{\pi}{12}$$

Substitution in definite integrals: examples

5. $\displaystyle\int_{x=0}^{x=1} xe^{x^2}\,dx \quad \begin{bmatrix} u = x^2 & \\ du = 2xdx \implies xdx = \dfrac{du}{2} & \\ x=0 \implies u=0 & \\ x=1 \implies u=1 & \end{bmatrix} = \int_{u=0}^{u=1} e^u\,\frac{du}{2} = \frac{1}{2}e^u\Big|_0^1 = \frac{e-1}{2}$

Here is an alternative way of expressing the calculation:

$$\int_{x=0}^{x=1} xe^{x^2}\,dx = \int_{x=0}^{x=1} d\left(\frac{1}{2}e^{x^2}\right) = \frac{1}{2}e^{x^2}\Big|_0^1 = \frac{e-1}{2}$$

6. $\displaystyle\int_{x=e}^{x=e^2} \frac{dx}{x\sqrt{\ln x}} \quad \begin{bmatrix} u = \ln x & \\ du = \dfrac{dx}{x} & \\ x=e \implies u = \ln e = 1 & \\ x=e^2 \implies u = \ln e^2 = 2 & \end{bmatrix} = \int_{u=1}^{u=2} \frac{du}{\sqrt{u}} = \int_{u=1}^{u=2} u^{-1/2}\,du =$

$$= 2u^{1/2}\Big|_1^2 = 2(\sqrt{2}-1)$$

Trigonometric integrals

Integrating trigonometric functions may require, besides techniques of integration, experience in working with trigonometric identities.

Here is a list of the trigonometric formulas which are used most often in integration problems:

$\sin^2 x + \cos^2 x = 1$

$\cos^2 x = \dfrac{1+\cos 2x}{2}$ $\qquad$ $\sin^2 x = \dfrac{1-\cos 2x}{2}$

$\dfrac{1}{\cos^2 x} = 1 + \tan^2 x$ $\qquad$ $\dfrac{1}{\sin^2 x} = 1 + \cot^2 x$

$2\sin x \cos x = \sin 2x$

$\sin x \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y))$

$\cos x \cos y = \frac{1}{2}(\cos(x+y) + \cos(x-y))$

$\sin x \cos y = \frac{1}{2}(\sin(x+y) + \sin(x-y))$

How to integrate even powers of sine and cosine

In integrals of type $\displaystyle\int (\cos^{2n} x)\, dx$ and $\displaystyle\int (\sin^{2n} x)\, dx$, where n is a positive integer,

use the half-angle formulas $\boxed{\cos^2 x = \dfrac{1+\cos 2x}{2}}$ and $\boxed{\sin^2 x = \dfrac{1-\cos 2x}{2}}$.

1. $\displaystyle\int \cos^2 x\, dx = ?$ ☞ Use the half-angle formula $\cos^2 x = \dfrac{1+\cos 2x}{2}$,

it converts the product $\cos x \cdot \cos x$ to a sum, which is easier to integrate:

$$\int \cos^2 x\, dx = \int \frac{1+\cos 2x}{2}\, dx = \frac{1}{2}\int (1+\cos 2x)dx = \frac{1}{2}(x + \frac{1}{2}\sin 2x) + C$$
$$= \frac{1}{2}x + \frac{1}{4}\sin 2x + C.$$

How to integrate even powers of sine and cosine

2. $\displaystyle\int \sin^4 x\,dx = \int (\sin^2 x)^2\,dx$ ☞ Use $\sin^2 x = \dfrac{1-\cos 2x}{2}$

$\displaystyle\int (\sin^2 x)^2\,dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$ ☞ Use $(a-b)^2 = a^2 - 2ab + b^2$

$\displaystyle\int \left(\frac{1-\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)dx$

$\displaystyle= \frac{1}{4}(x - \sin 2x) + \frac{1}{4}\int \cos^2 2x dx$ ☞ Use $\cos^2 2x = \dfrac{1+\cos 4x}{2}$

$$= \frac{1}{4}(x - \sin 2x) + \frac{1}{8}\int (1+\cos 4x)\,dx = \frac{1}{4}(x-\sin 2x) + \frac{1}{8}(x + \frac{1}{4}\sin 4x)$$

$$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

How to integrate odd powers of sine and cosine

1. $\displaystyle\int \cos^3 x dx = \int \cos^2 x \cdot \cos x\,dx = \int (1-\sin^2 x)\cdot \cos x\,dx$

$$\left[\begin{array}{l} u = \sin x \\ du = \cos x\,dx \end{array}\right] = \int (1-u^2)\,du = u - \frac{1}{3}u^3 + C = \sin x - \frac{1}{3}\sin^3 x + C.$$

2.

$$\int \sin^5 x dx = \int \sin^4 x \sin x\,dx = \int (\sin^2 x)^2 \sin x\,dx = \int (1-\cos^2 x)^2 \sin x\,dx$$

$$\left[\begin{array}{l} u = \cos x \\ du = -\sin x\,dx \end{array}\right] = \int (1-u^2)^2\,(-du) = -\int (1-2u^2+u^4)\,du$$

$$= -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + C = -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C$$

When both sine and cosine are present

1. $\int \sin^8 x \cos^3 x\,dx = \int \sin^8 x \cos^2 x \cos x\,dx = \int \sin^8 x(1-\sin^2 x)\cos x\,dx$

$\left[\begin{array}{l} u = \sin x \\ du = \cos x\,dx \end{array}\right] = \int u^8(1-u^2)\,du = \int (u^8 - u^{10})\,du = \frac{1}{9}u^9 - \frac{1}{11}u^{11} + C$

$= \frac{1}{9}\sin^9 x - \frac{1}{11}\sin^{11} x + C.$

2. $\int \sin(2x)\cos(3x)\,dx$ ☞ Use $\sin x \cos y = \frac{1}{2}(\sin(x+y) + \sin(x-y))$

$= \frac{1}{2}\int (\sin(2x+3x) + \sin(2x-3x))\,dx = \frac{1}{2}\int (\sin 5x + \sin(-x))\,dx$

$= \frac{1}{2}\int (\sin 5x - \sin x)\,dx = \frac{1}{2}\left(-\frac{1}{5}\cos(5x) + \cos x\right) + C$

$= -\frac{1}{10}\cos(5x) + \frac{1}{2}\cos x + C.$

Inverse trigonometric substitutions

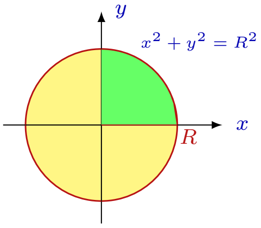
Integrals involving irrational expressions like $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$, $\sqrt{a^2+x^2}$

are calculated by special substitutions which are called *inverse trigonometric substitutions*.

Here is an example explaining the name “inverse trigonometric substitution”.

Problem. Use integrals to calculate the area of a disk of radius R.

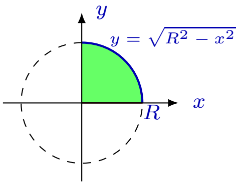
Solution.



Introduce a coordinate system in which the disk is centered at the origin. We have to calculate the area enclosed by the circle $x^2 + y^2 = R^2$.

By symmetry, the enclosed area is 4 times as large as the area of a quarter of the disk.
Let us calculate the area of this quarter.

The area of a quarter-disk



In the first quadrant (where $y \geq 0$),

$x^2 + y^2 = R^2 \implies y = \sqrt{R^2 - x^2}$.

The region is located below

the graph of the function $y = \sqrt{R^2 - x^2}$.

Therefore the area is $\displaystyle\int_0^R \sqrt{R^2 - x^2}\, dx$.

We calculate this integral using a special substitution. So far we have introduced a new variable u as a function of the original variable x .

In this case, we work the other way around:

we express the original variable x in terms of a new variable, which we call θ :

$x = R\sin\theta$. The reason for such substitution will become clear when we proceed with integration.

The inverse sine substitution

$$\int_0^R \sqrt{R^2 - x^2}\, dx \quad \begin{bmatrix} x = R\sin\theta \\ dx = R\cos\theta\, d\theta \\ x = 0 \implies \theta = 0 \\ x = R \implies \theta = \pi/2 \end{bmatrix}$$

since $x = R\sin\theta \iff \theta = \arcsin\dfrac{x}{R}$
this substitution is called
the *inverse trigonometric substitution*

$$= \int_0^{\pi/2} \sqrt{R^2 - R^2\sin^2\theta} \cdot R\cos\theta\, d\theta = R^2 \int_0^{\pi/2} \sqrt{1 - \sin^2\theta} \cdot \cos\theta\, d\theta$$

[Observe that $\cos\theta \geq 0$ in the first quadrant, so $\sqrt{1 - \sin^2\theta} = \cos\theta$]

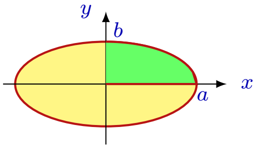
$$= R^2 \int_0^{\pi/2} \cos\theta \cdot \cos\theta\, d\theta = R^2 \int_0^{\pi/2} \frac{1 + \cos 2\theta}{2}\, d\theta = \frac{R^2}{2} \int_0^{\pi/2} (1 + \cos 2\theta)\, d\theta$$

$$= \frac{R^2}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Bigg|_0^{\pi/2} = \boxed{\frac{\pi R^2}{4}} = \text{ the area of a quarter of the disk.}$$

So the area of the disk of radius R is πR^2.

The area of an elliptic disk

In a similar manner we can calculate the area of an *elliptic disk* bounded by the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$:



The area $=4\times$ the area in the 1st quadrant.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad\Longrightarrow\quad y=b\sqrt{1-\frac{x^2}{a^2}} \;=\frac{b}{a}\sqrt{a^2-x^2}.$$

The region is located below the graph of the function $y=\dfrac{b}{a}\sqrt{a^2-x^2}$. Its' area is $\dfrac{b}{a}\displaystyle\int_0^a \sqrt{a^2-x^2}\,dx$

$$\left[\begin{array}{l} x=a\sin\theta \\ dx=a\cos\theta\,d\theta \\ x=0 \implies \theta=0 \\ x=a \implies \theta=\pi/2 \end{array}\right] =\frac{b}{a}\int_0^{\pi/2}\sqrt{a^2-a^2\sin^2\theta}\cdot a\cos\theta\,d\theta \;=ba\int_0^{\pi/2}\cos^2\theta\,d\theta \;=\frac{ab}{2}\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta$$

$$=\frac{ab}{2}\left(\theta+\frac{\sin 2\theta}{2}\right)\Bigg|_0^{\pi/2} \;=\frac{ab\pi}{4}.$$

The area of the elliptic disk is, therefore, $\boxed{\pi ab}$

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Integrals involving $\sqrt{x^2-a^2}$

☞ Integrals involving $\sqrt{x^2-a^2}$ can be simplified by the substitution $x=\dfrac{a}{\cos\theta}$.

Example. $\displaystyle\int \frac{dx}{x\sqrt{x^2-4}}$

Let $x=\dfrac{2}{\cos\theta}$. Then $dx=\dfrac{2\sin\theta}{\cos^2\theta}\,d\theta$ and

$$x\sqrt{x^2-4}=\frac{2}{\cos\theta}\sqrt{\frac{4}{\cos^2\theta}-4}=\frac{4}{\cos\theta}\sqrt{\frac{1-\cos^2\theta}{\cos^2\theta}}=\frac{4\sin\theta}{\cos^2\theta}.$$

Therefore,

$$\int\frac{dx}{x\sqrt{x^2-4}}=\int\frac{\frac{2\sin\theta}{\cos^2\theta}\,d\theta}{\frac{4\sin\theta}{\cos^2\theta}}=\frac{1}{2}\int d\theta=\frac{1}{2}\theta+C$$

What is θ in terms of x?

$$x=\frac{a}{\cos\theta}\implies x\cos\theta=a\implies \cos\theta=\frac{a}{x}\implies \theta=\arccos\frac{a}{x}.$$

Therefore, $\dfrac{1}{2}\theta+C=\boxed{\dfrac{1}{2}\arccos\dfrac{a}{x}+C}$

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Integrals involving $\sqrt{a^2+x^2}$ or $1/(a^2+x^2)$

☞ Integrals involving $\sqrt{a^2+x^2}$ or $\dfrac{1}{a^2+x^2}$

can be simplified by the substitution $x = a\tan\theta$.

Example. $\displaystyle\int \frac{dx}{x^2\sqrt{9+x^2}}$. Let $x = 3\tan\theta$, then $dx = \dfrac{3}{\cos^2\theta}\,d\theta$ and

$$\sqrt{9+x^2} = \sqrt{9+9\tan^2\theta} = 3\sqrt{1+\frac{\sin^2\theta}{\cos^2\theta}} = \sqrt{\frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}} = \frac{3}{\cos\theta}.$$

Therefore,

$$\int \frac{dx}{x^2\sqrt{9+x^2}} = \int \frac{\frac{3}{\cos^2\theta}\,d\theta}{9\tan^2\theta\frac{3}{\cos\theta}} = \frac{1}{9}\int \frac{\cos\theta}{\sin^2\theta}d\theta \; \left[\begin{array}{l} u = \sin\theta \\ du = \cos\theta\,d\theta \end{array}\right] = \frac{1}{9}\int \frac{du}{u^2}$$

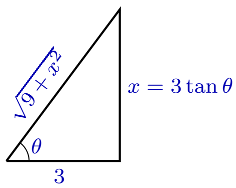
$$= -\frac{1}{9u} + C = -\frac{1}{9\sin\theta} + C.$$

How to express $\sin\theta$ in terms of x?

A helpful bit of right-triangle geometry

Recall that $x = 3\tan\theta$. It follows that $\theta = \arctan\dfrac{x}{3}$.

In this case, $\sin\theta = \sin\left(\arctan\dfrac{x}{3}\right)$. Here's how one can simplify this formula:



$$\sin\theta = \frac{x}{\sqrt{9+x^2}}$$

Therefore,

$$\int \frac{dx}{x^2\sqrt{9+x^2}} = -\frac{1}{9\sin\theta} + C = -\frac{1}{9\frac{x}{\sqrt{9+x^2}}} + C = \boxed{-\frac{\sqrt{9+x^2}}{9x} + C}$$

Inverse trigonometric substitutions: summary

integrand involves	$\sqrt{a^2 - x^2}$	$\sqrt{x^2 - a^2}$	$\sqrt{a^2 + x^2}$ or $\dfrac{1}{a^2 + x^2}$
substitution	$x = a \sin\theta$	$x = \dfrac{a}{\cos\theta}$	$x = a \tan\theta$

Summary

In this lecture, we learned how to calculate integrals by **substitution**.

Remember the substitution formula:

$$\int_{x=a}^{x=b} f(g(x))g'(x)\,dx = \int_{u=g(a)}^{u=g(b)} f(u)du$$

Some substitutions are obvious, some are not.

A substitution makes sense only if it **simplifies** the integral.

Comprehension checkpoint

- Calculate the following integrals:

$$\int_0^1 e^{3x}\,dx, \quad \int \frac{1}{1-2x}\,dx, \quad \int \sin^2 x\,dx, \quad \int_0^{\pi} \sin x \cos^2 x\,dx, \quad \int \frac{\sqrt{\ln x}}{x}\,dx.$$