Lecture 32

Applications of The Fundamental Theorem

Objectives

Remember that The Fundamental Theorem of Calculus (FTC) establishes
a **connection** between the definite and indefinite integrals:

If $f(x)$ is a **continuous** function on $[a, b]$, then

1) $\frac{d}{dx}\int_a^x f(t)dt = f(x)$

2) $\int_a^b f(x)dx = F(b) - F(a)$, where F is an antiderivative of f,

that is, any function F with $F'(x) = f(x)$.

In this lecture we will show how to apply the Fundamental Theorem of Calculus
to the differentiation of integrals and the calculation of definite integrals.

Differentiating an integral

Problem. Using the FTC, find the **derivatives** of the following integrals:

a) $\int_0^x e^{-t^2}dt$, **b)** $\int_x^2 \cos(t^2)\,dt$ **c)** $\int_1^{e^{3x}} \sqrt{t^2+t}\,dt.$

Solution. Observe that all the integrands are continuous functions,
so we may apply the first part of FTC: $\frac{d}{dx}\int_a^x f(t)dt = f(x)$.

a) $\frac{d}{dx}\int_0^x e^{-t^2}dt = e^{-x^2}$

b) $\frac{d}{dx}\int_x^1 \cos(t^2)\,dt = \frac{d}{dx}\left(-\int_1^x \cos(t^2)\,dt\right) = -\frac{d}{dx}\left(\int_1^x \cos(t^2)\,dt\right) = -\cos(x^2)$

c) $\frac{d}{dx}\int_1^{e^{3x}} \sqrt{t^2+t}\,dt \; [\, u = e^{3x} \,] = \frac{d}{dx}\int_1^u \sqrt{t^2+t}\,dt$ use the chain rule

$= \frac{d}{du}\left(\int_1^u \sqrt{t^2+t}\,dt\right)\cdot\frac{du}{dx} = \sqrt{u^2+u}\cdot 3e^{3x} = 3\sqrt{e^{6x}+e^{3x}}e^{3x} = 3\sqrt{e^{3x}+1}e^{9x/2}$

Differentiating an integral

Problem. Find $\dfrac{d}{dx}\displaystyle\int_{\sin x}^{\cos x}\frac{dt}{1-t^2}$.

Solution. The FTC can't be used directly since the integral to differentiate is not of the type $\displaystyle\int_a^x f(t)dt$.

To proceed, we split the integral into a **sum** of two integrals in each of which the variable x appears only once:

$$\int_{\sin x}^{\cos x}\frac{dt}{1-t^2}=\int_{\sin x}^{a}\frac{dt}{1-t^2}+\int_{a}^{\cos x}\frac{dt}{1-t^2}.$$

It will turn out that the choice of a does not matter.

Reverse the limits of integration in the first integral:

$$\int_{\sin x}^{a}\frac{dt}{1-t^2}=-\int_{a}^{\sin x}\frac{dt}{1-t^2}.$$

Now we are ready to differentiate:

$$\frac{d}{dx}\int_{\sin x}^{\cos x}\frac{dt}{1-t^2}=\frac{d}{dx}\left(-\int_{a}^{\sin x}\frac{dt}{1-t^2}+\int_{a}^{\cos x}\frac{dt}{1-t^2}\right)$$

4 / 21

Differentiating an integral

$$\frac{d}{dx}\int_{\sin x}^{\cos x}\frac{dt}{1-t^2}=\frac{d}{dx}\left(-\int_{a}^{\sin x}\frac{dt}{1-t^2}+\int_{a}^{\cos x}\frac{dt}{1-t^2}\right)$$

$$=-\frac{d}{dx}\int_{a}^{\sin x}\frac{dt}{1-t^2}+\frac{d}{dx}\int_{a}^{\cos x}\frac{dt}{1-t^2}$$

Use the chain rule

$$=-\left(\frac{1}{1-\sin^2 x}\right)\frac{d}{dx}\sin x+\left(\frac{1}{1-\cos^2 x}\right)\frac{d}{dx}\cos x$$

$$=-\frac{\cos x}{1-\sin^2 x}-\frac{\sin x}{1-\cos^2 x}=-\frac{\cos x}{\cos^2 x}-\frac{\sin x}{\sin^2 x}=-\frac{1}{\cos x}-\frac{1}{\sin x}.$$

Answer: $\dfrac{d}{dx}\displaystyle\int_{\sin x}^{\cos x}\frac{dt}{1-t^2}=-\frac{1}{\cos x}-\frac{1}{\sin x}$

5 / 21

Calculus for probability

In probability theory and statistics there is a very important function

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt\,,$$

which is called the cumulative distribution function for the normal distribution.

Let us consider a simplified version.

Problem. Let $f(x)=\displaystyle\int_0^x e^{-t^2}dt$.

a) Find the extreme points of f .

b) Find a linear approximation to f near $x=0$.

c) Find the inflection points of f .

Solution.
a) $f'(x)=\dfrac{d}{dx}\displaystyle\int_0^x e^{-t^2}dt=e^{-x^2}$.

Since $f'(x)\neq 0$, of f has **no** extreme points.

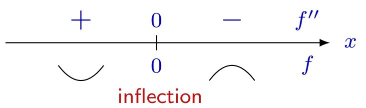
Calculus for probability

b) For the linear approximation $L(x)$ of $f(x)$ near $x=0$, we use the formula $f(x)\approx L(x)=f(0)+f'(0)(x-0)$. Since $f(0)=\displaystyle\int_0^0 e^{-t^2}dt=0$ and $f'(0)=e^{-x^2}\Big|_{x=0}=1$, we obtain $L(x)=0+1\cdot(x-0)=x$.

Therefore $\displaystyle\int_0^x e^{-t^2}dt\approx x$ near 0 .

c) For the inflection points of f , we study f'' :

$$f''(x)=\frac{d}{dx}e^{-x^2}=-2xe^{-x^2}\,.$$



The only inflection point of f is $x=0$.

Elementary integration

1. $\int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$. Intergration is the inverse of differentiation.

differentiate

2. $\int_0^1 2^x dx = \frac{2^x}{\ln 2}\Big|_0^1 = \frac{2^1}{\ln 2} - \frac{2^0}{\ln 2} = \frac{1}{\ln 2}$.

3. $\int_{-1}^1 \frac{dx}{1+x^2} = 2\int_0^1 \frac{dx}{1+x^2}$ since $f(x) = \frac{1}{1+x^2}$ is **even** and $[-1,1]$ is **symmetric**.

$= 2\arctan x\Big|_0^1 = 2(\arctan 1 - \arctan 0) = 2\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{2}$.

Elementary integration

4. $\int_{-1}^2 (x^3 - 2x + 1)dx = \left(\frac{x^4}{4} - x^2 + x\right)\Big|_{-1}^2$

$= \frac{2^4}{4} - 2^2 + 2 - \left(\frac{(-1)^4}{4} - (-1)^2 + (-1)\right) = 4 - 4 + 2 - \left(\frac{1}{4} - 1 - 1\right) = \frac{15}{4}$

5. $\int_1^8 \frac{\sqrt[3]{x^2} + 3x - 1}{x^2} dx = \int_1^8 \left(x^{\frac{2}{3}-2} + \frac{3}{x} - x^{-2}\right) dx = \int_1^8 \left(x^{-\frac{4}{3}} + \frac{3}{x} - x^{-2}\right) dx$

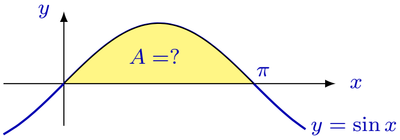
☞ $\int x^a\, dx = \frac{1}{a+1}x^{a+1}$ if $a \neq -1$ $\qquad \int \frac{dx}{x} = \ln|x|$

$= \left(\frac{1}{-\frac{4}{3}+1}x^{-\frac{4}{3}+1} + 3\ln|x| - \frac{1}{-2+1}x^{-2+1}\right)\Big|_1^8 = \left(-3x^{-\frac{1}{3}} + 3\ln|x| + x^{-1}\right)\Big|_1^8$

$-3\cdot 8^{-\frac{1}{3}} + 3\ln 8 + 8^{-1} - \left(-3\cdot 1^{-\frac{1}{3}} + 3\ln 1 + 1^{-1}\right) = -\frac{3}{2} + 9\ln 2 + \frac{1}{8} + 3 - 1 = \frac{5}{8} + 9\ln 2$

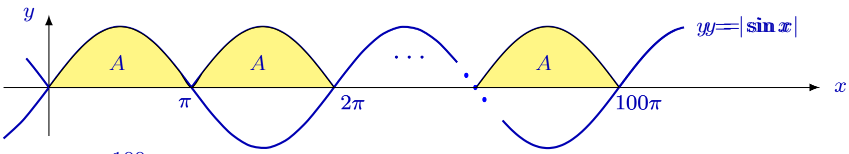
Area via the integral

Problem 1. Find the area of the region located between one arc of the sine curve and the x-axis.



$$A = \int_0^{\pi} \sin x\,dx \;= -\cos x\Big|_0^{\pi}$$
$$= -(\cos\pi - \cos 0) = -(-1-1) = \boxed{2}$$

Problem 2. Evaluate $\displaystyle\int_0^{100\pi} |\sin x|\,dx$.

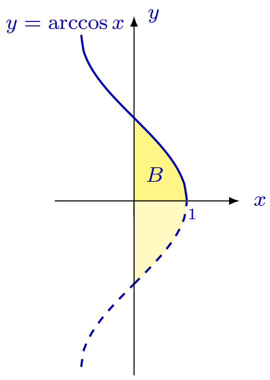


$$\int_0^{100\pi} |\sin x|\,dx = \;= 100A = 100\cdot 2 = \boxed{200}$$

The integral via area

Problem 3. Evaluate $\displaystyle\int_0^{1} \arccos x\,dx$.

The integral represents the area under the graph of $y = \arccos x$:



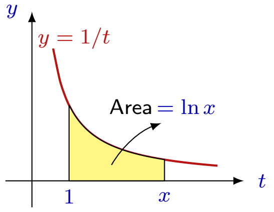
$B =$ a half of the area under one arc of the cosine curve
$= \frac{1}{2}\cdot 2 = 1$, see Problem 1.

Therefore, $\displaystyle\int_0^{1} \arccos x\,dx = \frac{1}{2}\cdot 2 = \boxed{1}$.

What is $\ln x$, really?

How would you explain $\ln x$ to your little brother or sister?

Draw the graph of $y = \dfrac{1}{t}$ for positive t:



Consider the region under the graph between the lines $t = 1$ and $t = x > 1$.

Its area is $\displaystyle\int_1^x \frac{1}{t}\,dt = \ln t\Big|_{t=1}^{t=x} = \ln x - \ln 1 = \ln x$

Therefore, $\ln x$ is the **area** of the region under the graph of the hyperbola.

Control question: How would you explain to your little brother or sister

$\ln x$ if $0 < x < 1$?

Warning

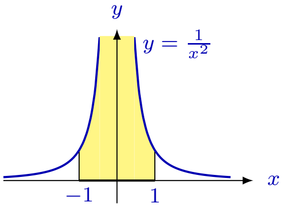
Problem. Evaluate the integral $\displaystyle\int_{-1}^{1} \frac{1}{x^2}dx$.

"Solution". $\displaystyle\int_{-1}^{1} \frac{1}{x^2}dx = -\frac{1}{x}\Big|_{-1}^{1} = -(1-(-1)) = -2$.

The answer doesn't seem to be plausible:

we integrated a **positive** function and obtained a **negative** result.

What went wrong? The integral represents the area under the curve $y = \dfrac{1}{x^2}$:

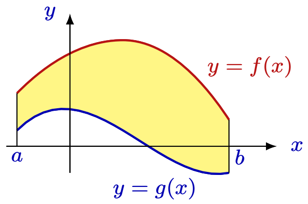


The region under the graph is not bounded.
What could its area be?
This topic will be discussed in Calculus II.

Applying of the FTC to this integral is **illegal**, since $f(x) = \dfrac{1}{x^2}$ is **not** continuous on $[-1, 1]$.

The area of the region between two curves

Theorem. Let $f(x), g(x)$ be continuous functions with $f(x) \geq g(x)$ on $[a, b]$.

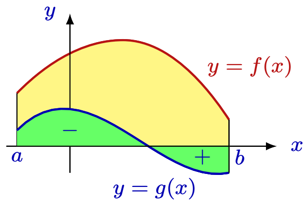


Then the area of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the vertical lines $x = a$, $x = b$ is given by the formula $\displaystyle\int_a^b (f(x) - g(x))dx.$

Proof. $\displaystyle\int_a^b (f(x) - g(x))dx = \int_a^b f(x)dx - \int_a^b g(x)dx =$

(signed area between f and the x-axis) $-$ (signed area between g and the x-axis)

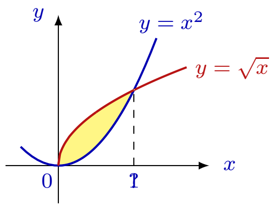
$=$ area between f and g.



The area between two curves

Example 1. Find the area of the region bounded by $y = x^2$ and $y = \sqrt{x}$.

Solution.



Fist, find the intersection points of the curves:

$x^2 = \sqrt{x} \implies x^4 = x \implies x(x^3 - 1) = 0$

$\implies x(x-1)(x^2 + x + 1) = 0$

$\implies x = 0$ or $x = 1.$

The area between the **upper** curve $y = \sqrt{x}$ and the **lower** curve $y = x^2$ is

$$\int_0^1 (\sqrt{x} - x^2)dx = \frac{2}{3}x^{3/2} - \frac{x^3}{3}\Big|_0^1 = \frac{2}{3} - \frac{1}{3} = \boxed{\frac{1}{3}}$$

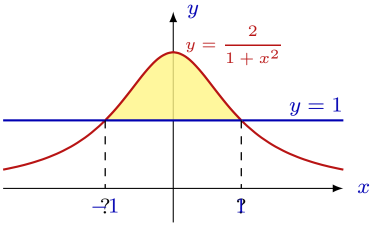
Answer: the area of the region is $1/3$ square units.

☞ The area of a region is always **non-negative**.

The area between two curves

Example 2. Find the area of the region situated below the curve $y = \dfrac{2}{1+x^2}$ and above the line $y = 1$.

Solution.



What are the intersection points of $y = \dfrac{2}{1+x^2}$ and $y = 1$?

$\dfrac{2}{1+x^2} = 1 \iff 2 = 1 + x^2 \iff x = \pm 1$

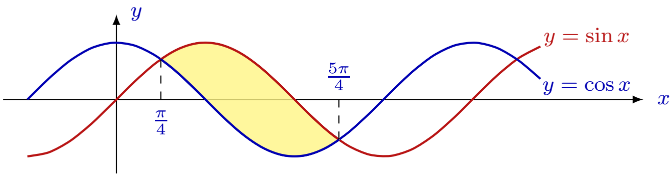
The area between the curves is $\displaystyle\int_{-1}^{1} \left(\frac{2}{1+x^2} - 1 \right) dx = 2 \int_{0}^{1} \left(\frac{2}{1+x^2} - 1 \right) dx = 4 \int_{0}^{1} \frac{dx}{1+x^2} - 2 \int_{0}^{1} dx$

$= 4 \arctan x \Big|_{x=0}^{x=1} - 2 = 4(\arctan 1 - \arctan 0) - 2 = 4 \cdot \dfrac{\pi}{4} - 2 = \boxed{\pi - 2}$

The area between the waves

Example 3. Find the area of one of the regions between the sine and cosine curves.

Solution. Draw a picture:



Find the intersection points of the curves over one period:

$\sin x = \cos x \implies x = \pi/4$ and $x = 5\pi/4$.

The area between the waves is

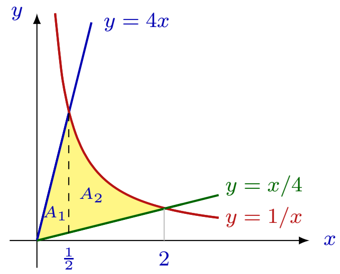
$\displaystyle\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\, dx = (-\cos x - \sin x)\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} = -(\cos x + \sin x)\Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} = -\left(\cos \tfrac{5\pi}{4} + \sin \tfrac{5\pi}{4}\right) + \left(\cos \tfrac{\pi}{4} + \sin \tfrac{\pi}{4}\right)$

$= -\left(-\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2}\right) + \left(\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}\right) = \boxed{2\sqrt{2}}$.

The area bounded by three curves

Example 4. Find the area of the region bounded by $y = \frac{1}{x}$, $y = 4x$ and $y = \frac{x}{4}$ in the 1st quadrant.

Solution.



This region is **not** between two curves.
Split it into two regions A_1 and A_2 which are.
Find the limits of integration:

$$\frac{1}{x} = 4x \iff 4x^2 = 1 \implies x = 1/2,$$

$$\frac{1}{x} = \frac{x}{4} \iff x^2 = 4 \implies x = 2.$$

The area bounded by these three curves is the sum of areas of A_1 and A_2:

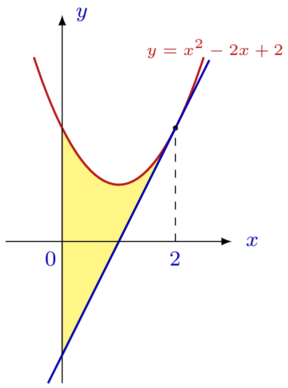
$$\int_0^{1/2} (4x - x/4)\,dx + \int_{1/2}^{2} (1/x - x/4)\,dx = \Big[2x^2 - x^2/8\Big]_0^{1/2} + \Big[\ln x - x^2/8\Big]_{1/2}^{2}$$

$$\frac{1}{2} - \frac{1}{32} + \ln 2 - \frac{1}{2} - \ln\frac{1}{2} + \frac{1}{32} = \ln 2 + \ln 2 = \boxed{2\ln 2}$$

The area between a curve and its tangent

Example 5. Find the area of the region bounded by
the parabola $y = x^2 - 2x + 2$, its tangent line at $x = 2$, and the y-axis.

Solution. Draw a picture:



The equation of the tangent line is $y - y(2) = y'(2)(x-2)$.
Since $y' = 2x - 2$, we have $y'(2) = 2$,
and the equation is $y - 2 = 2(x-2) \iff y = 2x - 2$.

The area of the region is $\int_0^2 (x^2 - 2x + 2 - (2x-2))\,dx$

$$= \int_0^2 (x^2 - 4x + 4)\,dx = \int_0^2 (x-2)^2\,dx$$

$$= \frac{1}{3}(x-2)^3 \Big|_0^2 = \boxed{\frac{8}{3}}$$

Summary

In this lecture we demonstrated some important applications of the Fundamental Theorem of Calculus.

Comprehension checkpoint

- Let $f(x) = \int_1^x \frac{\ln t}{t}\, dt$. Find $\frac{d}{dx} f(x)$.
- Evaluate the following integrals:

$$\int_{-\pi}^{\pi/2} \sin x\, dx, \quad \int_0^1 x^2 + 2^x\, dx, \quad \int_0^1 \frac{x}{x^2+x}\, dx$$

- Find the area of the region bounded by the curve $y = e^x$, its tangent line at $x = 0$ and the vertical line $x = 1$.