Lecture 31

The Fundamental Theorem of Calculus

Objectives

In this lecture we will discuss the Fundamental Theorem of Calculus (FTC) which establishes a **connection** between the definite and indefinite integrals.

We will **formulate** the theorem and give a sketch of a **proof**.

Dummy variable

As we know, the definite integral $\int_a^b f(x)dx$ represents the **signed area** of the region between the graph of $f(x)$, the x-axis and the vertical lines $x = a$, $x = b$.

In this integral notation, the variable x appears as a placeholder.
The value of the integral does not depend on how we call the variable.

It is called a **dummy variable** and may be changed to any other variable unless it would lead to confusion:

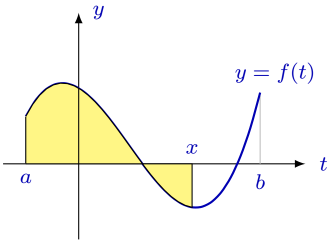
$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(s)ds = \int_a^b f(y)dy \text{ etc.}$$

A similar phenomenon of dummy variable may be observed in summations:

$$\sum_{i=1}^{n} i = \sum_{j=1}^{n} j = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

The area-so-far function

Let $f(t)$ be a continuous function on $[a, b]$:



Consider a new function $A(x) = \int_a^x f(t)dt$.

$A(x)$ is the signed area of the region between the graph of the function, the x-axis, and two vertical lines.

We call $A(x)$ the "area-so-far" function.

$A(a) = \int_a^a f(t)dt = 0$

$A(b) = \int_a^b f(t)dt = \int_a^b f(x)dx$

What is the **rate of change** of $A(x)$? That is, $\frac{dA}{dx} =?$

Spoiler: $\frac{dA}{dx} = f(x)$.

4 / 13

FTC: statement

Theorem (The Fundamental Theorem of Calculus).

Let $f(x)$ be a **continuous** function on $[a, b]$. Then

1) $\frac{d}{dx} \int_a^x f(t)dt = f(x)$

2) $\int_a^b f(x)dx = F(b) - F(a)$, where F is an antiderivative of f, that is, any function with $F'(x) = f(x)$.

5 / 13

FTC: discussion

Remarks. Why is this theorem called **fundamental**?

1. It establishes a **connection** between the two main operations in calculus,
differentiation and integration:

$$\int_a^x f(t)dt \xrightarrow[\text{differentiate}]{\frac{d}{dx}} f(x)$$

$$F'(x) \xrightarrow[\text{integrate}]{\int_a^b} F(b) - F(a)$$

Differentiation and integration are **inverse** processes:
what the derivative does, the definite integral undoes;
what the definite integral does, the derivative undoes.

FTC: discussion

2. The second part of FTC, namely $\int_a^b f(x)dx = F(b) - F(a)$, is called *the evaluation theorem* or *the Newton-Leibniz theorem*.

It may be written out in different ways. Since $F'(x)dx = \frac{dF}{dx}dx = dF$, we may write

$$\int_a^b f(x)dx = \int_a^b F'(x)dx = \int_a^b \frac{dF}{dx}\,dx = \int_a^b dF\,.$$

Therefore, $\int_a^b f(x)dx = F(b) - F(a) \iff \int_a^b dF = F(b) - F(a)\,.$

It is convenient to use the following **evaluation** notation: $F(b) - F(a) = F(x)\Big|_a^b$
Then the evaluation theorem takes the following form

$$\int_a^b f(x)dx = F(x)\Big|_a^b \quad \text{or} \quad \int_a^b dF = F(x)\Big|_a^b$$

FTC: discussion

3. The evaluation theorem $\int_a^b f(x)dx = F(x)\Big|_a^b$ establishes a **connection** between the definite integral $\int_a^b f(x)dx$ and the indefinite integral $F(x)$.

It gives the **primary tool** for calculation of definite integrals.

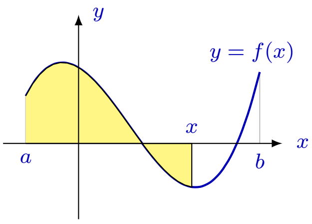
Other tools for computing integrals (signed areas or limits of Riemann sums) can't compete in efficiency with the evaluation theorem.

8 / 13

FTC: proof

1) We have to prove that if $f(x)$ is a **continuous** function on $[a, b]$, then

$$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$



Let $A(x) = \int_a^x f(t)dt$ be the area-so-far function.

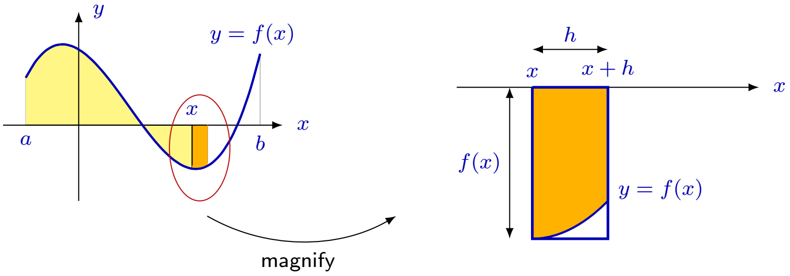
We have to show that $A(x)$ is differentiable and $A'(x) = f(x)$.

By the definition of the derivative, $A'(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h}$.

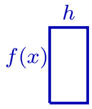
Let us study the area increment $A(x+h) - A(x)$.

9 / 13

FTC: proof



The area increment $A(x+h) - A(x)$ is the signed area of the region.

$A(x+h) - A(x) \approx$ the signed area of the rectangle 

$= hf(x)$
with the approximation getting better as $h \to 0$.

Therefore, $A'(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h\to 0} \frac{hf(x)}{h} = f(x)$.

FTC: proof

2) We need to prove that $\int_a^b f(x)dx = F(x)\Big|_a^b$ for any antiderivative F of f.

We know from **1)** that A is an antiderivative of f since $A'(x) = f(x)$.

Any two antiderivatives differ by a constant: $A(x) = F(x) + C$.

By the definition of A, $A(x) = \int_a^x f(t)dt$.

So $A(b) = \int_a^b f(t)dt$ and $A(a) = \int_a^a f(t)dt = 0$.

Therefore,

$$\int_a^b f(x)dx = A(b) = A(b) - \underbrace{A(a)}_{0} = F(b) + C - (F(a) + C) = F(b) - F(a),$$ as required.

Summary

In this lecture, we stated and proved the Fundamental Theorem of Calculus:

If $f(x)$ is a **continuous** function on $[a,b]$, then

1) $\frac{d}{dx}\int_a^x f(t)dt = f(x)$

2) $\int_a^b f(x)dx = F(b) - F(a)$, where F is an antiderivative of f,

that is, any function F with $F'(x) = f(x)$.

Comprehension checkpoint

- State the Fundamental Theorem of Calculus.
- What does this theorem say for $f(x) = x^2$ and $[a,b] = [0,1]$?

Give a geometric illustration for the theorem in this case.