Lecture 30

Riemann Sums. Part 2

Objectives

In this lecture we will continue to work with the definite integral as the limit of Riemann sums.

Remember that for a piece-wise continuous function $f(x)$, the definite integral
may be calculated as the limit of a Riemann sum L_n or R_n:

$\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n,$ where

$L_n = \sum\limits_{i=0}^{n-1} f(x_i)\Delta x$ is the left Riemann sum,

$R_n = \sum\limits_{i=1}^{n} f(x_i)\Delta x$ is the right Riemann sum,

$\Delta x = \dfrac{b-a}{n}$ are n subintervals of $[a, b]$,

$x_i = a + i\Delta x$ are the points subdividing $[a, b]$.

Riemann sums have a simple geometric interpretation.
In this lecture, we will do some elementary calculations with Riemann sums.

The area under a parabola

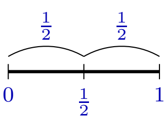
Problem. 1. Approximate $\displaystyle\int_0^1 x^2 dx$ by L_2 and R_2.

2. Calculate the Riemann sums L_n and R_n, where n is a positive integer.

3. Calculate $\displaystyle\int_0^1 x^2 dx$ explicitly as the limit of L_n and R_n.

Solution. The integrand is $f(x) = x^2$, the interval is $[0, 1]$.

1. To calculate L_2 and R_2, we subdivide $[0, 1]$ into **two** equal parts of length $\Delta x = \dfrac{b-a}{n} = \dfrac{1}{2}$:

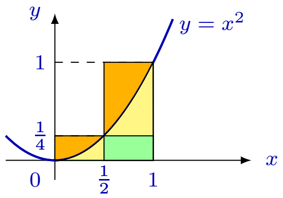


The subdivision points are

$x_0 = 0,\ x_1 = \dfrac{1}{2},\ x_2 = 1$.

The area under a parabola

The integral $\int_0^1 x^2 dx$ represents the area under the parabola $y = x^2$ between $x = 0$ and $x = 1$:



$$L_2 = f(x_0)\Delta x + f(x_1)\Delta x \; = f(0)\cdot\frac{1}{2} + f(\tfrac{1}{2})\cdot\frac{1}{2} \; = 0\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{1}{2} = \boxed{\frac{1}{8}}.$$

$$R_2 = f(x_1)\Delta x + f(x_2)\Delta x \; = f(\tfrac{1}{2})\cdot\frac{1}{2} + f(1)\cdot\frac{1}{2} \; = \frac{1}{4}\cdot\frac{1}{2} + 1\cdot\frac{1}{2} = \boxed{\frac{5}{8}}.$$

$\int_0^1 x^2 dx \approx L_2 = \frac{1}{8}$ (underestimate), $\int_0^1 x^2 dx \approx R_2 = \frac{5}{8}$ (overestimate)

The area under a parabola

2. To calculate the general Riemann sums L_n, R_n,
we subdivide $[0,1]$ into n equal parts of length $\Delta x = \frac{b-a}{n} = \frac{1}{n}$.

The points of the subdivision are $x_i = x_0 + i\Delta x = 0 + i\frac{1}{n} = \frac{i}{n}$, where $i = 0, 1, \ldots, n$.

$$L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x = \sum_{i=0}^{n-1} f\left(\frac{i}{n}\right)\frac{i}{n} = \sum_{i=0}^{n-1}\left(\frac{i}{n}\right)^2\frac{1}{n} = \frac{1}{n^3}\sum_{i=0}^{n-1} i^2,$$

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\frac{i}{n} = \sum_{i=1}^{n}\left(\frac{i}{n}\right)^2\frac{1}{n} = \frac{1}{n^3}\sum_{i=1}^{n} i^2.$$

As we know from Algebra (or can prove by induction),

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

The area under a parabola

Therefore,

$$L_n = \frac{1}{n^3}\sum_{i=0}^{n-1} i^2 = \frac{(n-1)n(2n-1)}{6n^3} \quad \text{and}$$

$$R_n = \frac{1}{n^3}\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6n^3}.$$

The Riemann sums L_n and R_n approximate $\int_0^1 x^2dx$ and,
since $f(x) = x^2$ is increasing on $[0,1]$, we have

$L_n \le \int_0^1 x^2dx \le R_n$ for **any** positive integer n. Calculate the limits:

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} \frac{(n-1)n(2n-1)}{6n^3} = \lim_{n\to\infty} \frac{(1-\frac{1}{n})\cdot 1\cdot(2-\frac{1}{n})}{6} = \frac{1\cdot 1\cdot 2}{6} = \frac{1}{3},$$

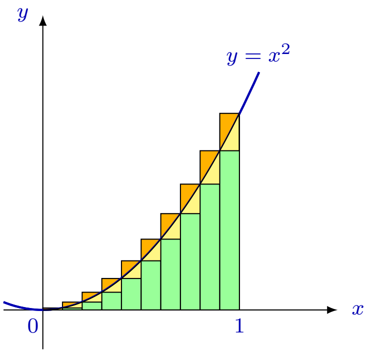
$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} = \lim_{n\to\infty} \frac{1\cdot(1+\frac{1}{n})\cdot(2+\frac{1}{n})}{6} = \frac{1\cdot 1\cdot 2}{6} = \frac{1}{3}.$$

The area under a parabola

In the inequality for the integral, let $n \to \infty$:

$$\begin{array}{ccccc} \boxed{L_n} & \le & \int_0^1 x^2dx & \le & \boxed{R_n} \\ & \searrow^{n\to\infty} & \| & \swarrow^{n\to\infty} & \\ & & \frac{1}{3} & & \end{array}$$

Therefore, $\int_0^1 x^2dx = \frac{1}{3}.$



$$\int_0^1 x^2dx = \lim_{n\to\infty} \boxed{L_n} = \frac{1}{3}$$

$$\int_0^1 x^2dx = \lim_{n\to\infty} \boxed{R_n} = \frac{1}{3}$$

Approximating an Integral by Riemann sums

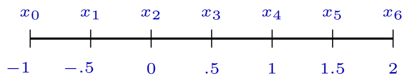
Problem. 1. Approximate $\int_{-1}^{2}(x^3-2x)dx$ by L_6 and R_6.

2. Calculate the Riemann sums L_n and R_n, where n is a positive integer.

3. Calculate $\int_{-1}^{2}(x^3-2x)dx$ as the limit of L_n and R_n.

Solution. The integrand is $f(x)=x^3-2x$, the interval is $[-1,2]$.

1. To calculate L_6 and R_6, we subdivide $[-1,2]$ into **six** equal parts of length $\Delta x=\dfrac{b-a}{n}=\dfrac{2-(-1)}{6}=\dfrac{3}{6}=\dfrac{1}{2}$:



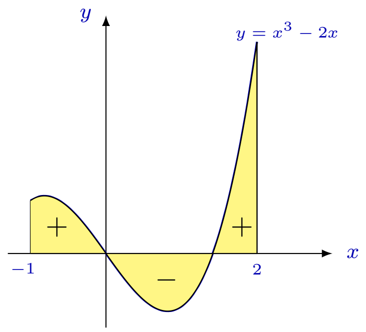
The subdivision points are

$x_0=-1,\ x_1=-.5,\ x_2=0,\ x_3=.5,\ x_4=1,\ x_5=1.5,\ x_6=2.$

In abbreviated form, $x_i=x_0+i\Delta x=-1+\dfrac{i}{2}$, where $i=0,1,\ldots,6$.

The integral as a signed area

The integral $\int_{-1}^{2}(x^3-2x)dx$ represents the **signed** area between the graph of $y=x^3-2x$ and the $x-$ axis:

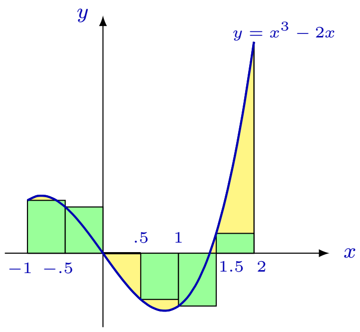


The geometric interpretation of the left Riemann sum

$$\boxed{L_6} = \sum_{i=0}^{5} f(x_i)\Delta x$$

$$= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x$$

$$= \Big(f(-1) + f(-.5) + f(0) + f(.5) + f(1) + f(1.5)\Big) \cdot \frac{1}{2} = \boxed{\frac{3}{16}}$$

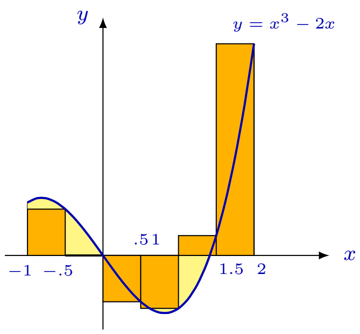


The geometric interpretation of the right Riemann sum

$$\boxed{R_6} = \sum_{i=1}^{6} f(x_i)\Delta x$$

$$= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5) + f(x_6)\Delta x$$

$$= \Big(f(-.5) + f(0) + f(.5) + f(1) + f(1.5) + f(2)\Big) \cdot \frac{1}{2} = \boxed{\frac{27}{16}}$$



Left and right Riemann sums

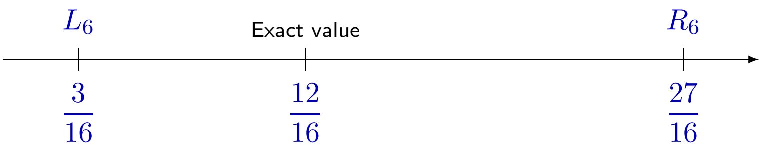
We have calculated $L_6 = \dfrac{3}{16}$ and $R_6 = \dfrac{27}{16}$.

Both left and right Riemann sums approximate the value of the integral:

$$\int_{-1}^{3} (x^3 - 2x)dx \approx \frac{3}{16} \quad \text{and} \quad \int_{-1}^{3} (x^3 - 2x)dx \approx \frac{27}{16}.$$

The exact value of the integral, as we will calculate soon, is

$$\int_{-1}^{2} (x^3 - 2x)dx = \frac{3}{4} = \frac{12}{16}.$$



General left and right Riemann sums

To compute L_n and R_n for $\displaystyle\int_{-1}^{2} (x^3 - 2x)dx$, recall the formulas:

$$L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x \quad \text{and} \quad R_n = \sum_{i=1}^{n} f(x_i)\Delta x, \text{ where}$$

$$\Delta x = \frac{2-(-1)}{n} = \frac{3}{n} \quad \text{and} \quad x_i = x_0 + i\Delta x = -1 + \frac{3i}{n}, \quad i = 0, 1, 2, \ldots, n.$$

Since $f(x_i) = f\left(-1 + \dfrac{3i}{n}\right) = \left(-1 + \dfrac{3i}{n}\right)^3 - 2\left(-1 + \dfrac{3i}{n}\right)$, we find

$$L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x = \sum_{i=0}^{n-1} \left[\left(-1 + \frac{3i}{n}\right)^3 - 2\left(-1 + \frac{3i}{n}\right)\right]\frac{3}{n} \text{ and}$$

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \left[\left(-1 + \frac{3i}{n}\right)^3 - 2\left(-1 + \frac{3i}{n}\right)\right]\frac{3}{n}.$$

General left and right Riemann sums

After algebraic manipulations, we get

$$L_n = \frac{3}{n}\sum_{i=0}^{n-1}\left(1+\frac{3}{n}i-\frac{27}{n^2}i^2+\frac{27}{n^3}i^3\right) = \frac{3}{n}\sum_{i=0}^{n-1}1+\frac{9}{n^2}\sum_{i=0}^{n-1}i-\frac{81}{n^3}\sum_{i=0}^{n-1}i^2+\frac{81}{n^4}\sum_{i=0}^{n-1}i^3$$

$$\stackrel{(*)}{=} \frac{3}{n}n+\frac{9}{n^2}\frac{(n-1)n}{2}-\frac{81}{n^3}\frac{(n-1)n(2n-1)}{6}+\frac{81}{n^4}\left(\frac{(n-1)n}{2}\right)^2$$

$$= 3+\frac{9}{2}\left(1-\frac{1}{n}\right)-\frac{27}{2}\left(1-\frac{1}{n}\right)1\left(2-\frac{1}{n}\right)+\frac{81}{4}\left(\left(1-\frac{1}{n}\right)1\right)^2.$$

So $\lim_{n\to\infty} L_n = 3+\frac{9}{2}-27+\frac{81}{4}=\frac{3}{4}$.

Similarly, $R_n = \frac{3}{n}\sum_{i=1}^{n}\left(1+\frac{3}{n}i-\frac{27}{n^2}i^2+\frac{27}{n^3}i^3\right)$, and $\lim_{n\to\infty} R_n \stackrel{(*)}{=} \frac{3}{4}$.

In our calculations (*), we used the following formulas:

$$\sum_{i=1}^{n}1=n,\quad \sum_{i=1}^{n}i=\frac{n(n+1)}{2},\quad \sum_{i=1}^{n}i^2=\frac{n(n+1)(2n+1)}{6},\quad \sum_{i=1}^{n}i^3=\left(\frac{n(n+1)}{2}\right)^2$$

The integral as the limit of Riemann sums

Since $\int_a^b f(x)dx = \lim_{n\to\infty} L_n$ or $\int_a^b f(x)dx = \lim_{n\to\infty} R_n$, we have

$$\int_{-1}^{2}(x^3-2x)dx = \lim_{n\to\infty}\frac{3}{n}\sum_{i=0}^{n-1}\left(1+\frac{3}{n}i-\frac{27}{n^2}i^2+\frac{27}{n^3}i^3\right)=\frac{3}{4} \text{ or}$$

$$\int_{-1}^{2}(x^3-2x)dx = \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left(1+\frac{3}{n}i-\frac{27}{n^2}i^2+\frac{27}{n^3}i^3\right)=\frac{3}{4}.$$

Finally, $\int_{-1}^{2}(x^3-2x)dx = \frac{3}{4}$.

☞ The calculation of definite integrals using Riemann sums is tedious.

A more efficient method of calculation (using the Fundamental Theorem of Calculus) will be given in the next lecture.

Calculation of limits

Example. Calculate the limit $\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{n-i}{n^2}$.

Solution. We calculate the limit by interpreting it as the limit of a Riemann sum, and expressing the limit of the Riemann sum as an integral.

The sum $\sum_{i=0}^{n-1} \frac{n-i}{n^2}$ looks like the **left** Riemann sum $\sum_{i=0}^{n-1} f(x_i)\Delta x$.

That would make $\frac{n-i}{n^2} = f(x_i)\Delta x$, which we can write as $\left(1 - \frac{i}{n}\right)\frac{1}{n} = f(x_i)\Delta x$. This gives $\Delta x = \frac{b-a}{n} = \frac{1}{n}$, $x_i = \frac{i}{n}$, $f(x_i) = 1 - x_i$.

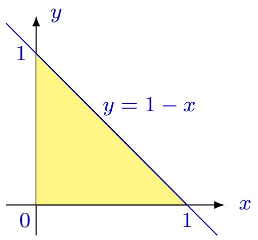
It follows that $b - a = 1$, $a = x_0 = \frac{0}{n} = 0$, $f(x) = 1 - x$.

Putting it all together, $a = 0, b = 1, f(x) = 1 - x$ and

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{n-i}{n^2} = \int_0^1 (1-x)dx .$$

Calculation of limits

Now we evaluate the integral $\int_0^1 (1-x)dx$.



$$\int_0^1 (1-x)dx = \text{Area of triangle} = \frac{1}{2}$$

Finally, $\lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{n-i}{n^2} = \int_0^1 (1-x)dx = \boxed{\frac{1}{2}}$.

Summary

Riemann sums may be used for the approximate calculation of definite integrals.

Remember the following formulas:

$\int_a^b f(x)\,dx = \lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n,$ where

$L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x$ is the left Riemann sum,

$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$ is the right Riemann sum,

$\Delta x = \dfrac{b-a}{n}$ are subintervals,

$x_i = a + i\Delta x$ are the subdivision points of $[a, b]$.

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Comprehension checkpoint

- Approximate the integral $\int_0^1 e^{x^2}\,dx$ by the Riemann sums L_2, R_2 and L_4, R_4. Use a calculator!

Give geometric interpretations of these Riemann sums.

Are the approximations underestimates or overestimates?

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