Lecture 29

Riemann Sums. Part 1

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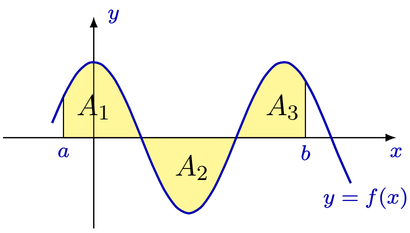
Objectives

In this lecture we will discuss an approach to the definite integral as the limit of Riemann sums.

Bernhard Riemann (1826-1866)

The definite integral as a signed area

The *definite integral* of a piece-wise continuous function was defined as
the **signed area** of the region bounded by
the graph of f, the x-axis, and the lines $x = a, x = b$.
The area above the x-axis adds to the total, the area below the x-axis subtracts from the total.



$$\int_a^b f(x)dx = A_1 - A_2 + A_3$$

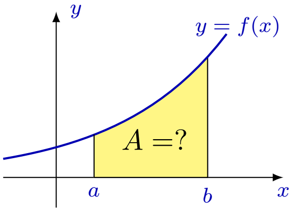
To calculate the integrals, we have to be able to calculate the areas.

So far, we did this only for very simple functions,
where the area can be calculated by tools from elementary geometry.

How do we calculate the area for more general functions?

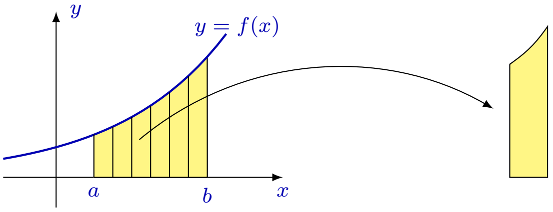
The area problem

Problem. Given a piece-wise continuous function $f(x)$ and numbers a, b.



How to calculate $A = \int_a^b f(x)dx$ (the signed area)?

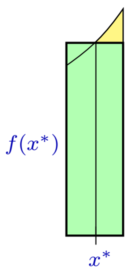
$\int_a^b f(x)dx$ is the sum of the areas of thin *curvilinear trapezoids*:



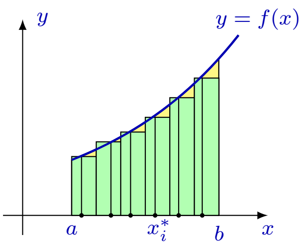
Area=?

Curvilinear trapezoids

How to calculate the area of a curvilinear trapezoid?



Its area can be approximated by the area of a rectangle with the same base and height $f(x^*)$, where x^* is an arbitrary point in the base.



$$\int_a^b f(x)\,dx \approx \sum_i \underbrace{f(x_i^*)}_{\text{height}} \underbrace{\Delta x_i}_{\text{base}}$$

Riemann sums

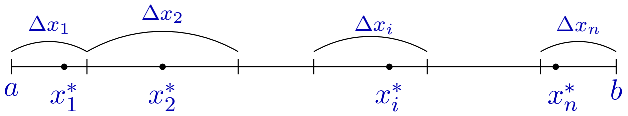
An expression of type $\sum\limits_i f(x_i^*)\Delta x_i$ is called a *Riemann sum* for the integral $\int\limits_a^b f(x)\,dx$.

Remarks.

1. The larger the number n of rectangles, the better the approximation

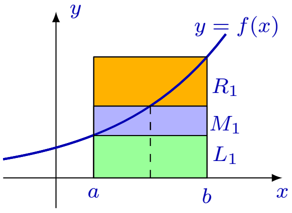
$$\int\limits_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

2. Δx_i are the bases of the rectangles, $x_i^* \in \Delta x_i$ are arbitrary points:



Riemann sums for small numbers of rectangles

Let us see how the integral is approximated by Riemann sums with a small number of rectangles.
By **one** rectangle:



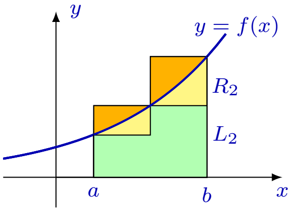
$A \approx L_1$, the base point x_1^* is the left end point of $[a, b]$
$A \approx R_1$, the base point x_1^* is the right end point of $[a, b]$
$A \approx M_1$, the base point x_1^* is the middle point of $[a, b]$

Since the function in this example is increasing, we have $L_1 \le A \le R_1$

It's not a good approximation, but a good start!

Approximation by two rectangles

We approximate area under the graph by **two** rectangles:



$A \approx L_2$

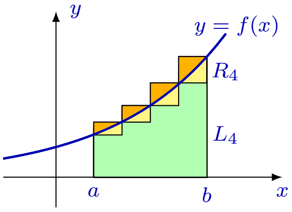
$A \approx R_2$

$A \approx M_2$

$L_1 \le \boxed{L_2 \le A \le R_2} \le R_1$

for an increasing function f

One can increase the number of approximating rectangles:



$A \approx L_4$

$A \approx R_4$

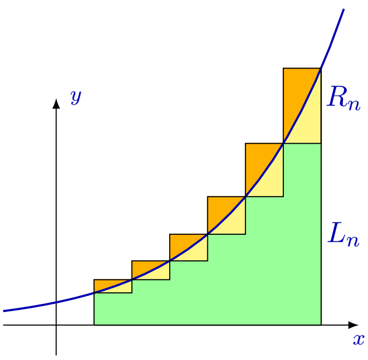
$A \approx M_4$

$L_1 \le L_2 \le L_3 \le \boxed{L_4 \le A \le R_4} \le R_3 \le R_2 \le R_1$

for an increasing function f

The larger the number of rectangles, the better the approximation!

Approximation by n rectangles



L_n is the sum of the areas of left rectangles

R_n is the sum of the areas of right rectangles

For an increasing f, we have

$L_1 \le \cdots \le L_n \le A \le R_n \cdots \le R_1$

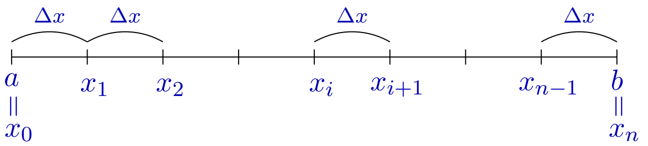
$n \to \infty$ $\searrow$ A $\swarrow$ $n \to \infty$

$$\int_a^b f(x)dx = A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n = \lim_{n\to\infty} M_n$$

Riemann sums

Construction of Riemann sums: subintervals

Let us formalize the idea of calculation of the definite integral as the limit of special sums.
Let $f(x)$ be a piecewise continuous function (not necessarily positive) on the interval $[a, b]$.
Fix a positive integer n and subdivide $[a, b]$ into n **equal** parts:



Call the points of the subdivision $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$.

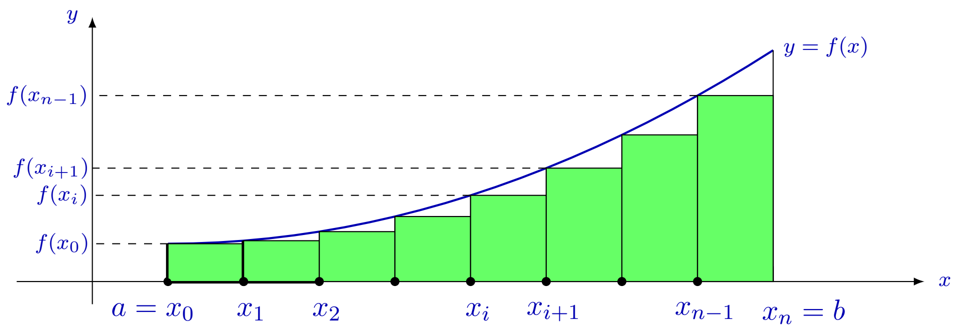
The interval $[a, b]$ is subdivided into n intervals of the same length $\Delta x = \dfrac{b-a}{n}$.

The coordinates x_i can be expressed in terms of a, b and n:

$x_i = x_0 + i\,\Delta x = a + i\,\dfrac{b-a}{n}$ for each $i = 0, 1, \ldots, n$.

Construction of Riemann sums: left rectangles

From each x_i, draw the vertical segment from the x-axis to the graph:



Over each subinterval $[x_i, x_{i+1}]$, construct a rectangle of height $f(x_i)$,
the value of the function at the **left endpoint** of the subinterval.

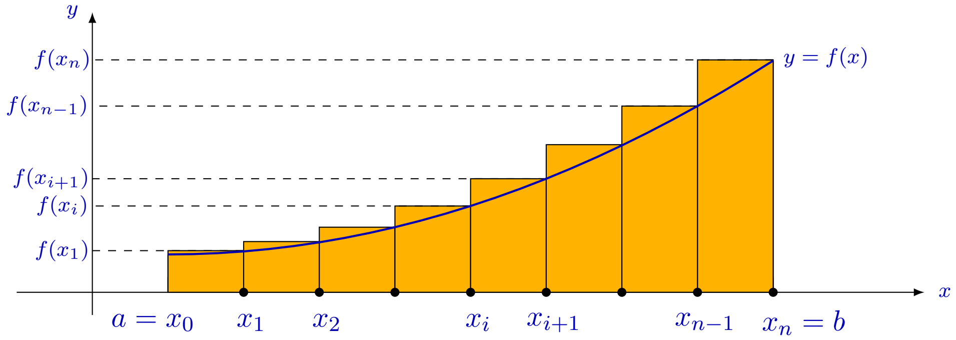
The **left Riemann sum** is

$L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=0}^{n-1} f(x_i)\Delta x$

If $f(x) \geq 0$, then L_n is the area of a polygon approximating the area under the graph.

Construction of Riemann sums: right rectangles

The **right Riemann sum** is constructed in a similar way:



Over each subinterval $[x_i, x_{i+1}]$, construct a rectangle of height $f(x_{i+1})$, the value of the function at the **right endpoint** of the subinterval.

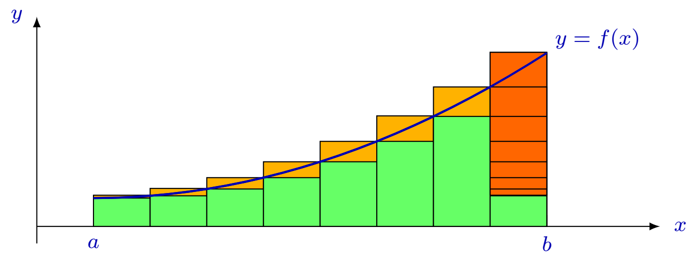
The **right Riemann sum** is $R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x.$

When the integral is squeezed between Riemann sums

For an increasing function, like the one in our picture,
the area under the graph is squeezed between the polygonal areas:

$$L_n \leq \int_a^b f(x)\,dx \leq R_n$$

Notice that $R_n - L_n = (f(b) - f(a))\Delta x = (f(b) - f(a))\dfrac{b-a}{n} \underset{n\to\infty}{\longrightarrow} 0:$



The integral is the limit of Riemann sums

One can prove that both limits $\lim_{n\to\infty} L_n$ and $\lim_{n\to\infty} R_n$ exist, and, since for an increasing function we have

$$L_n \le \int_a^b f(x)\,dx \le R_n\,,$$

the Squeeze theorem guarantees that in this case $\int_a^b f(x)\,dx = \lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n.$

One can prove that the same holds true for **any** function bounded on $[a,b]$:

$\int_a^b f(x)\,dx = \lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n,$ where L_n and R_n are left and right Riemann sums.

In a similar way one can construct a general Riemann sum $R(f,n)$
for any function f that is bounded on the interval $[a,b]$
by subdividing the interval into n subintervals Δx_i of **arbitrary** lengths, and choosing **arbitrary** points $x_i^* \in \Delta x_i$. In this case, as long as $\max\Delta x_i \to 0,$

the limit still exists and is equal to the integral: $\int_a^b f(x)\,dx = \lim_{\substack{n \to \infty \\ \max\Delta x_i \to 0}} R(f,n)$

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Summary

In this lecture we learned

- how to express a definite integral as the limit of Riemann sum.
- how to approximate a definite integral by Riemann sums.

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Comprehension checkpoint

- Approximate the integral $\displaystyle\int_{-2}^{2}(x+1)\,dx$ by the Riemann sums L_4 and R_4.

Give geometric interpretations to these Riemann sums.

What is the exact value of the integral?

Present the integral as the limit of R_n.