Lecture 28

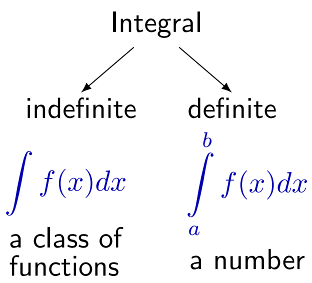
The Definite Integral

Objectives

In this lecture we will discuss the definition and properties of the **definite integral**.

Indefinite and definite integrals

Recall that in calculus there are two kinds of integrals: **indefinite** and **definite**:



The indefinite and definite integrals are **related** by the Fundamental Theorem of Calculus, which we will study soon.

☞ Don't confuse indefinite and definite integrals!

The definite integral

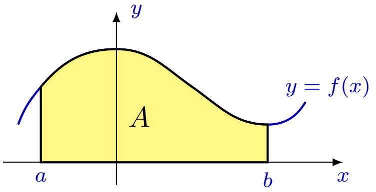
The notion of the definite integral can be explained using the notion of **area** of a plane figure.

To each plane figure, we associate a real number, called its area, which satisfies four properties called the **axioms of area**: monotonicity, additivity, invariance and normalization.

Understanding the definite integral as a special area helps
to operate easily with properties of the definite integral.

The definite integral as area

Let a, b be real numbers, $a \le b$, and $f(x)$ be a non-negative continuous function defined on $[a, b]$.



Let A be the **area** of the region
below the graph $y = f(x)$,
above the x-axis,
and between the lines $x = a$, $x = b$.

Definition. The *definite integral of* $f(x)$ *over the interval* $[a, b]$ is

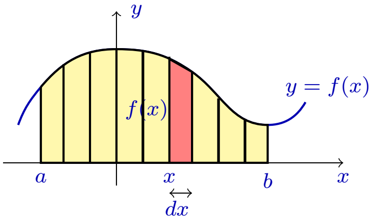
$$\int_a^b f(x)dx = A$$

$f(x)$ is called the *integrand*,
a, b are called the *limits* of integration,
dx is called the *differential*.

Integral notation

Why does the integral sign $\int$, which we have already used for the general antiderivative, appear here?

The integral sign $\int$ is a "designer's" version of the letter "S" standing for a sum. But which sum?



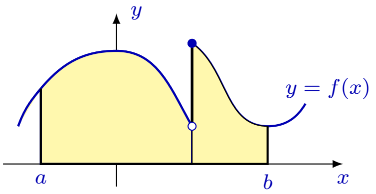
$f(x)dx$ is the **area** of a narrow strip.
It is called the *area element*
The total area A below the graph is obtained when we
sum up infinitely many infinitely "thin" areas.

$A =$ Area under the graph $=$ the infinite sum of "thin" areas $= \int_a^b f(x)dx$

The definite integral: is continuity important?

For a **non-negative continuous** function, the definite integral is defined as the **area** under the graph.

Is the continuity important? Not really, the definition makes sense also for a **piecewise continuous** and **bounded** function:

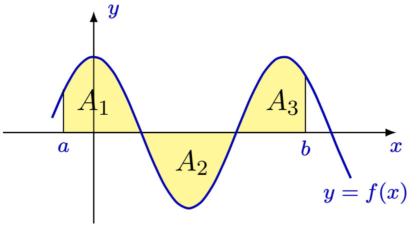


$\int_a^b f(x)dx =$ Area under the graph.

Is non-negativity important? Not really, the definition will be adjusted to work for functions taking positive and negative values.

The definite integral for a non-positive function

Let f be a piece-wise continuous function.



$$\int_a^b f(x)dx = A_1 - A_2 + A_3$$

Definition. The *definite integral of $f(x)$ over the interval $[a, b]$*,

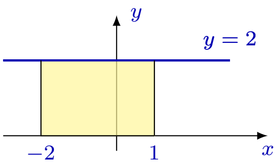
written $\int_a^b f(x)dx,$

is the **signed** area of the region bounded
by the graph of f, the x-axis, and the lines $x = a$, $x = b$.
The area **above** the x-axis adds to the total, the area below the x-axis **subtracts** from the total.

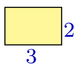
Evaluation of integrals as signed areas

Example 1. Evaluate the integral $\int_{-2}^{1} 2\,dx$ interpreting it as a **signed area**.

Solution. Draw the graph of the integrand $f(x) = 2$:



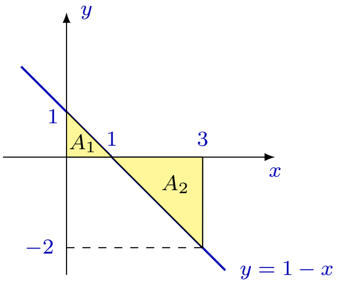
Draw the lines $x = -2$ and $x = 1$.
Determine the region with signed area
given by the integral.

$\int_{-2}^{1} 2\,dx =$ Area of () $= 6$.

Evaluation of integrals as signed areas

Example 2. Evaluate the integral $\int_0^3 (1-x)\,dx$ interpreting it as a **signed area**.

Solution. Draw the region between the graph of $y = 1 - x$, the x-axis, and the lines $x = 0$, $x = 3$:



$$\int_0^3 (1-x)\,dx = A_1 - A_2$$

$$= \frac{1}{2}\cdot 1 \cdot 1 - \frac{1}{2}\cdot 2 \cdot 2 = \boxed{-\frac{3}{2}}$$

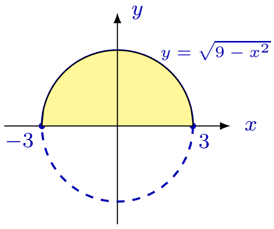
☞ An area is always **non-negative**, a definite integral may be **any** number.

Evaluation of integrals as signed areas

Example 3. Evaluate the integral $\int_{-3}^{3} \sqrt{9-x^2}\,dx$ interpreting it as a **signed area**.

Solution. To draw the graph of $y = \sqrt{9-x^2}$, observe that $y^2 = 9 - x^2$, that is $x^2 + y^2 = 9$.

Since $-3 \le x \le 3$ and $y \ge 0$, the graph of $y = \sqrt{9-x^2}$ is the upper half of the circle $x^2 + y^2 = 9$.



The integral represents the area between the graph and the x-axis:

$$\int_{-3}^{3} \sqrt{9-x^2}\,dx = \frac{1}{2}\pi 3^2 = \boxed{\frac{9\pi}{2}}$$

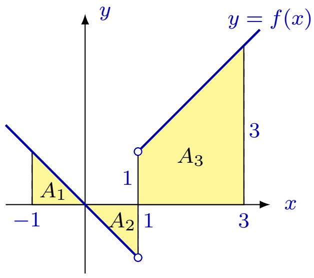
The integral of a piecewise continuous function

Example 4. Evaluate the integral $\displaystyle\int_{-1}^{3} \frac{x^2 - x}{|x-1|}\,dx$.

Solution. First, we simplify the integrand:

$$f(x) = \frac{x^2 - x}{|x-1|} = \frac{x(x-1)}{|x-1|} = \begin{cases} x, \text{ if } x > 1 \\ -x, \text{ if } x < 1. \end{cases}$$

Note that $f(x)$ is not defined at $x = 1$.



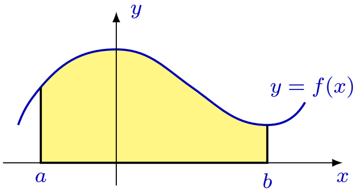
Indicate the limits of integration $x = -1$, $x = 3$.
Determine the area represented by the integral.

$$\int_{-1}^{3} \frac{x^2 - x}{|x-1|}\,dx = A_1 - A_2 + A_3 = A_3 = \frac{1+3}{2} \cdot 2 = \boxed{4}$$

Properties of the definite integral

Properties of the definite integral follow from the properties of area.

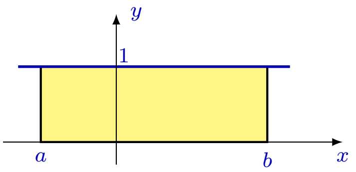
1. The integral measures area
If $f \geq 0$ is a piecewise continuous function on $[a, b]$, then



$\displaystyle\int_a^b f(x)dx =$ the area of the region

below the graph $y = f(x)$,
above the x -axis,
and between the lines $x = a,\ x = b$.

2. The integral can measure length (only works when integrand is 1)



$$\int_a^b dx = b - a$$

Properties of the definite integral

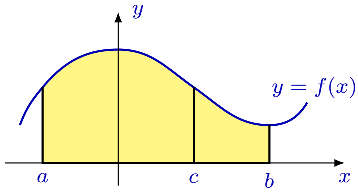
3. Linearity with respect to the integrand

$$\int_a^b \big(f(x)+g(x)\big)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$\int_a^b Cf(x)dx = C\int_a^b f(x)dx$, where C is a constant

4. Additivity with respect to the interval on integration

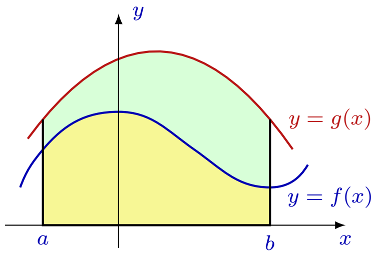
$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$



Properties of the definite integral

5. Monotonicity

If $f(x) \le g(x)$ on $[a,b]$, then $\int_a^b f(x)dx \le \int_a^b g(x)dx$



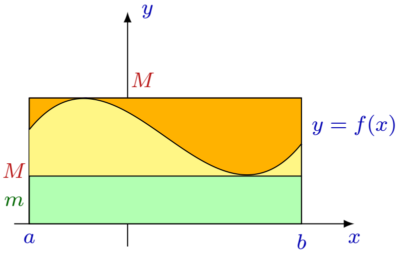
In particular, if $f(x) \ge 0$ on $[a,b]$, then $\int_a^b f(x)dx \ge 0$.

Properties of the definite integral

5. Lower and upper bounds

If $m \leq f(x) \leq M$ on $[a, b]$ for some constants m and M, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$



$m(b-a)$ is the area of the lower rectangle

$M(b-a)$ is the area of the upper rectangle

$\int_a^b f(x)dx$ is the area of the region under the graph.

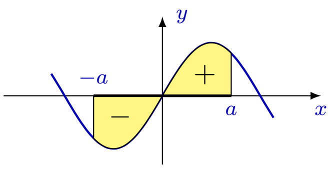
The inequality $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$ shows that the area of the region under the graph is somewhere between the areas of the rectangles.

Properties of the definite integral

6. Symmetry

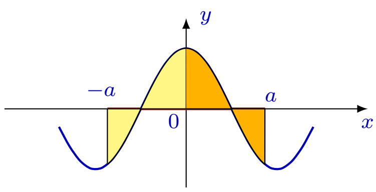
- If f is **odd** (that is $f(-x) = -f(x)$ for all x) and the interval of integration is **symmetric** about the origin, then

$$\int_{-a}^a f(x)dx = 0$$



- If f is **even** (that is $f(-x) = f(x)$ for all x) and the interval of integration is **symmetric** about the origin, then

$$\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$$



Properties of the definite integral

6. Integration over an interval of **zero length** gives zero:

$$\int_a^a f(x)dx = 0$$

7. Reversing the limits of integration reserves the sign of the integral:

If $b < a$ then $\displaystyle\int_a^b f(x)dx = -\int_b^a f(x)dx$

Indeed, $\displaystyle 0 = \int_b^b f(x)dx = \int_b^a f(x)dx + \int_a^b f(x)dx\,.$

So $\displaystyle\int_a^b f(x)dx = -\int_b^a f(x)dx\,.$

Use the properties!

Example 1. Evaluate the integral $\displaystyle\int_{-2}^{2}(3+5x)dx\,.$

Solution. $\displaystyle\int_{-2}^{2}(3+5x)dx = \int_{-2}^{2} 3dx + 5\int_{-2}^{2} xdx$ (by linearity)

$= 3 \cdot \underbrace{(2-(-2))}_{\text{length of } [-2,2]} + 5 \cdot 0$ (since $f(x) = x$ is **odd** and the interval is **symmetric**)

$= 3 \cdot 4 = \boxed{12}$

Example 2. Evaluate the integral $\displaystyle\int_{-\pi}^{\pi}\frac{\sin^3 x}{1+x^2}\,dx\,.$

Solution. The integrand $f(x) = \dfrac{\sin^3 x}{1+x^2}$ is **odd**:

$f(-x) = \dfrac{\sin^3(-x)}{1+(-x)^2} = \dfrac{(-\sin x)^3}{1+x^2} = -\dfrac{\sin x^3}{1+x^2} = -f(x)$ and

the interval $[-\pi,\pi]$ is **symmetric**. Therefore, $\displaystyle\int_{-\pi}^{\pi}\frac{\sin^3 x}{1+x^2}\,dx = \boxed{0}$.

Use the properties!

Example 3. Find lower and upper bounds for $\int_0^1 e^{-x^2} dx$.

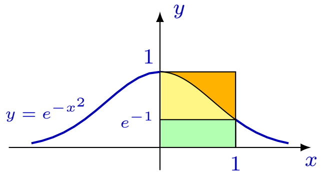
Solution. We have to **estimate** the integral from above and below

$$\underbrace{c}_{\text{lower bound}} \leq \int_0^1 e^{-x^2} dx \leq \underbrace{C}_{\text{upper bound}}$$

by finding a **lower bound** c and an **upper bound** C .

Use the fact that e^{-x^2} is monotonic to find c and C .
Since $f(x) = e^{-x^2}$ is decreasing on $[0, 1]$ ($f'(x) = -2xe^{-x^2} < 0$ for $x > 0$),
$e^{-1} = f(1) < e^{-x^2} < f(0) = 1$ for any $x \in [0, 1]$. Therefore,

$e^{-1} \cdot (1 - 0) \leq \int_0^1 e^{-x^2} dx \leq 1 \cdot (1 - 0)$, that is $\boxed{e^{-1}} \leq \int_0^1 e^{-x^2} dx \leq \boxed{1}$.



Summary

In this lecture, we studied

- the definite integral of a function over an intervalas a **signed area** of a special region
- properties of the definite integral
- evaluation techniques for the definite integral based on understanding the integral as a signed area.

Comprehension checkpoint

- Let $f(x) = \begin{cases} 1, & \text{if } x < 0 \\ x - 1, & \text{if } x \geq 0. \end{cases}$ Calculate $\int_{-2}^{4} f(x)\,dx$
- Without calculation of the integrals, explain why $\int_0^1 e^{x^2}\,dx > \int_0^1 \arctan x\,dx$
- Evaluate the integral $\int_{-1}^{1} \frac{x^5 + x}{\cos x}\,dx$