Lecture 27

Areas of Plane Figures

Objectives

This lecture opens our next topic, the **definite integral**.

Today we discuss what the **area** of a plane figure is
and how to calculate the area of a disc using the axioms of area.

Indefinite and definite integrals

In calculus, there are two kinds of integrals: **indefinite** and **definite**:

$$\begin{tikzcd} & \text{Integral} \arrow[dl] \arrow[dr] & \\ \text{indefinite} & & \text{definite} \end{tikzcd}$$

indefinite	definite
$\int f(x)dx$	$\int_a^b f(x)dx$
a class of functions	a number

The indefinite and definite integrals are **related** by the Fundamental Theorem of Calculus.

☞ Don't confuse indefinite and definite integrals!

The definite integral

The notion of the definite integral can be explained using the idea of the **area** of a plane figure.

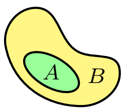
What is the area and how do we calculate it?

Area possesses a few remarkable properties which entirely determine it.

To each plane figure, we associate a real number, called its *area*, which satisfies several properties called the **axioms of area**.

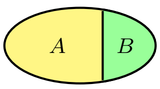
The axioms of area

1. Monotonicity:



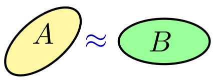
$A \subseteq B \implies \text{Area}(A) \leq \text{Area}(B)$

2. Additivity: if A, B have no inner points in common, then



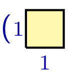
$\text{Area}(A \cup B) = \text{Area}(A) + \text{Area}(B)$

3. Invariance: if A is congruent to B, then



$\text{Area}(A) = \text{Area}(B)$

4. Normalization:

Area ( $) = 1$ square unit

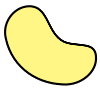
How to calculate the area of a figure

Axioms 1 and 2 ensure that the area of any figure is **non-negative**.

Indeed, for any A,
$\text{Area}(A) = \text{Area}(A \cup \varnothing) = \text{Area}(A) + \text{Area}(\varnothing) \implies \text{Area}(\varnothing) = 0$.
$\varnothing \subseteq A \implies \underbrace{\text{Area}(\varnothing)}_{0} \leq \text{Area}(A)$. So $\text{Area}(A) \geq 0$.

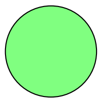
Exercise. Using Axioms 2-4, find the area of a rectangle, triangle, and parallelogram.

How to calculate the area of more complicated figures?



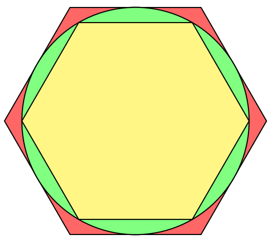
Area=?

Let us calculate the area of a disc of radius R. This calculation will give us a basic idea for area calculations.



Area=?

The area of a disk: idea of calculation



Let A be the area of the disk,

I the area of **inscribed** polygon,

S the area of **circumscribed** polygon.

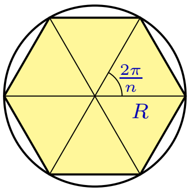
By monotonicity, $I \leq A \leq S$.

One can increase the number of sides of the inscribed and circumscribed polygons,
so I will increase and S will decrease.
By this, one can make the difference $S - I$ as **small** as possible.
The area of the disk is a **unique** number A such that $I \leq A \leq S$ for any I, S.

☞ The area of any plane figure can be calculated in a similar way,
via approximation by the areas of **inscribed** and **circumscribed** polygons .

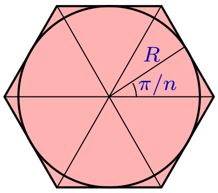
The area of a disk: inscribed and circumscribed polygons

Given a disk of radius R, let us inscribe in it a regular n-gon:



The area of the inscribed polygon is

$$I = n \cdot \frac{1}{2} R^2 \sin \frac{2\pi}{n}$$



The area of the circumscribed polygon is

$$S = n \cdot R^2 \tan \frac{\pi}{n}$$

The area of a disk: calculation

Since $I \leq A \leq S$, we get

$n \cdot \frac{1}{2} R^2 \sin \frac{2\pi}{n} \leq A \leq n \cdot R^2 \tan \frac{\pi}{n}$ or, equivalently,

$$\pi R^2 \frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}} \leq A \leq \pi R^2 \frac{\tan \frac{\pi}{n}}{\frac{\pi}{n}}$$

What happens if n (the number of the sides) grows to infinity? Both polygons are getting closer and closer to the disk.

Since $\lim\limits_{n\to\infty} \frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}} = \lim\limits_{x\to 0} \frac{\sin x}{x} = 1$, and $\lim\limits_{n\to\infty} \frac{\tan \frac{\pi}{n}}{\frac{\pi}{n}} = \lim\limits_{x\to 0} \frac{\tan x}{x} = 1$

we find as $n \to \infty$

$$\underbrace{\pi R^2 \frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}}}_{\substack{\downarrow \\ \pi R^2}} \leq A \leq \underbrace{\pi R^2 \frac{\tan \frac{\pi}{n}}{\frac{\pi}{n}}}_{\substack{\downarrow \\ \pi R^2}}$$

Therefore, A, the area of a disk of radius R, equals πR^2.

Summary/Comprehension checkpoint

In this lecture we have learned four **axioms** of area.
List these axioms.