Lecture 26

Elementary Integration

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Objectives

In this lecture, we discuss how to **calculate** indefinite integrals
using the table of antiderivatives and properties of antiderivatives.

We discuss applications of the indefinite integral to differential equations and kinematics.

Functions and their antiderivatives

For calculation of indefinite integrals, we need to remember
the table of antiderivatives. Keep in mind: $\underset{\text{function}}{f(x)} \xleftarrow{\frac{d}{dx}} \underset{\text{antiderivative}}{F(x)}$

$f(x)$	$F(x)$
x^a	$\dfrac{x^{a+1}}{a+1}$
$\dfrac{1}{x}$	$\ln\lvert x\rvert$
e^x	e^x
a^x	$\dfrac{a^x}{\ln a}$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$

$f(x)$	$F(x)$
$\dfrac{1}{\cos^2 x}$	$\tan x$
$\dfrac{1}{\sin^2 x}$	$-\cot x$
$\dfrac{1}{1+x^2}$	$\arctan x$
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin x$
$-\dfrac{1}{\sqrt{1-x^2}}$	$\arccos x$

Linearity of the anitiderivative

Also, for integration we will need to use the **linearity** of the antiderivative:

$$\int (f(x)+g(x))dx = \int f(x)dx + \int g(x)dx$$

$\int Cf(x)dx = C\int f(x)dx$, where C is a constant.

Integrating power functions

Example 1. Calculate $\int 4x^2\,dx$ and check your answer by differentiation.

Solution. $\int 4x^2\,dx$ (use linearity to move the constant outside the integral)

$= 4\int x^2\,dx$ (integrate a **power** function: $\int x^a dx = \frac{x^{a+1}}{a+1} + C$)

$= 4\,\frac{x^{2+1}}{2+1} + C = \frac{4}{3}x^3 + C.$

Check: $\frac{d}{dx}\left(\frac{4}{3}x^3 + C\right) = \frac{4}{3}\cdot 3\,x^2 = 4x^2$ ✓

☞ We can always **check** whether an antiderivative has been calculated correctly!

Example 2. Calculate $\int \sqrt[3]{x^2}\,dx$ and check your answer by differentiation.

Solution. $\int \sqrt[3]{x^2}\,dx = \int x^{\frac{2}{3}}\,dx = \frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} + C = \frac{3}{5}x^{\frac{5}{3}} + C\,.$

Check: $\frac{d}{dx}\left(\frac{3}{5}x^{\frac{5}{3}} + C\right) = \frac{3}{5}\cdot\frac{5}{3}x^{\frac{5}{3}-1} = x^{\frac{2}{3}} = \sqrt[3]{x^2}$ ✓

Integrating power functions

Example 3. Calculate $\displaystyle\int \left(\frac{3}{x^2} + \sqrt{x}\right)\,dx$ and check your answer by differentiation.

Solution. $\displaystyle\int \left(\frac{3}{x^2} + \sqrt{x}\right)\,dx = 3\int x^{-2}\,dx + \int x^{1/2}\,dx$ (by linearity)

$\displaystyle = 3\frac{x^{-2+1}}{-2+1} + \frac{x^{1/2+1}}{1/2+1} + C$ (integrate **power** function: $\displaystyle\int x^a dx = \frac{x^{a+1}}{a+1} + C$)

$\displaystyle = -3x^{-1} + \frac{x^{3/2}}{3/2} + C = -3x^{-1} + \frac{2}{3}x^{3/2} + C\,.$

Check: $\displaystyle\frac{d}{dx}\left(-3x^{-1} + \frac{2}{3}x^{3/2} + C\right) = 3x^{-2} + \frac{2}{3}\cdot\frac{3}{2}x^{1/2} = \frac{3}{x^2} + \sqrt{x}$ ✓

Integrating fractional functions

Example 4. Calculate $\displaystyle\int \frac{x^2+1}{x}\,dx$ and check your answer by differentiation.

Solution. $\displaystyle\int \frac{x^2+1}{x}\,dx = \int \left(x + \frac{1}{x}\right)\,dx = \frac{x^2}{2} + \ln|x| + C\,.$

Check: $\displaystyle\frac{d}{dx}\left(\frac{x^2}{2} + \ln|x| + C\right) = x + \frac{1}{x} = \frac{x^2+1}{x}$ ✓

Example 5. Calculate $\displaystyle\int \frac{x^2+4}{x^2+1}\,dx$

Solution. $\displaystyle\int \frac{x^2+4}{x^2+1}\,dx = \int \frac{(x^2+1)+3}{x^2+1}\,dx = \int \left(1 + \frac{3}{x^2+1}\right)\,dx$

$\displaystyle = \int 1\,dx + 3\int \frac{1}{x^2+1}\,dx = x + 3\arctan x + C\,.$

Integrating trigonometric functions

Example 6. Calculate $\int (2\sin x + 3\cos x)\, dx$ and check your answer by differentiation.

Solution.

$$\int (2\sin x + 3\cos x)\, dx = 2\int \sin x\, dx + 3\int \cos x\, dx = -2\cos x + 3\sin x + C$$

Check:

$$\frac{d}{dx}(-2\cos x + 3\sin x + C) = -2\frac{d}{dx}\cos x + 3\frac{d}{dx}\sin x = 2\sin x + 3\cos x \quad \checkmark$$

Integrating trigonometric functions

Example 7. Calculate $\int \cos^2 \frac{x}{2}\, dx$ and check your answer by differentiation

Solution. Integration trigonometric functions often requires the use of **trigonometric formulas**. In this case, we need the half-angle formula:

$$\boxed{\cos^2 \frac{x}{2} = \frac{1+\cos x}{2}}$$

$$\int \cos^2 \frac{x}{2}\, dx = \int \frac{1+\cos x}{2}\, dx = \frac{1}{2}\int (1+\cos x) dx = \frac{1}{2}\left(\int dx + \int \cos x\, dx\right)$$

$$= \frac{1}{2}(x + \sin x) + C = \frac{x+\sin x}{2} + C$$

Check: $\frac{d}{dx}\left(\frac{x+\sin x}{2} + C\right) = \frac{1+\cos x}{2} = \cos^2 \frac{x}{2} \quad \checkmark$

Differential equations

A *differential equation* (DE) is an equation involving derivatives of an unknown function.

For example, $y'' = 2x$ is a differential equation. It has infinitely many solutions:
any function $y(x) = \frac{1}{3}x^3 + C_1x + C_2$, where C_1, C_2 are constants,
is a solution. Check this!

The **general solution** of a differential equation contains **all** the solutions.

Problem. Find the general solution for the differential equation $y'' = \sin x$.

Solution. We have to find **all** functions $y = y(x)$ satisfying the equation.

First, we find y': $y'(x) = \int y''(x)dx = \int \sin x\, dx = -\cos x + C_1$, $C_1 \in \mathbb{R}$.

Then we find y: $y(x) = \int y'(x)dx = \int (-\cos x + C_1)dx = -\sin x + C_1x + C_2$, $C_2 \in \mathbb{R}$.

Answer. The general solution is $y(x) = -\sin x + C_1x + C_2$, where $C_1, C_2 \in \mathbb{R}$.

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Initial value problems for differential equations

An *initial value problem* (IVP) for a differential equation consists of

• a differential equation and

• the initial condition (prescribed values for the solution which are enough to determine the values of the constants introduced during integration).

Example. Solve the initial value problem and check your solution:

$$\begin{cases} y' = e^{2x} \\ y(0) = 1. \end{cases}$$

Solution. The IVP consists of the differential equation $y' = e^{2x}$ and the initial condition $y(0) = 1$.

Our task is to find a function $y = y(x)$ that satisfies the equation and the initial condition.

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Solving the DE

First, we find all functions $y(x)$ satisfying the differential equation $y' = e^{2x}$.

For this, we integrate both sides of the equation:

$\int y'(x)\,dx = \int e^{2x}\,dx$ and obtain

$y(x) = \frac{1}{2}e^{2x} + C$, where C is an arbitrary constant.

Keep in mind that it is enough to add an arbitrary constant to **one** side of the integrated equation.
To determine the value of C, we use the initial condition $y(0) = 1$.

Substitute $x = 0$ into the solution $y(x) = \frac{1}{2}e^{2x} + C$ of the DE:

$$y(0) = \frac{1}{2}e^{2x} + C\Big|_{x=0} = \frac{1}{2}e^{2\cdot 0} + C = \frac{1}{2}e^{0} + C = \frac{1}{2} + C.$$

Since $y(0) = 1$, we find $\frac{1}{2} + C = 1 \implies C = \frac{1}{2}$.

How to check the solution of an IVP

We have found $y(x) = \frac{1}{2}e^{2x} + C$ and $C = \frac{1}{2}$.

Therefore, the solution for the IVP is $\boxed{y(x) = \frac{1}{2}e^{2x} + \frac{1}{2}}$.

Let us **check** that the solution above satisfies the differential equation and the initial condition.

$$y'(x) = \frac{d}{dx}\left(\frac{1}{2}e^{2x} + \frac{1}{2}\right) = \frac{1}{2}2e^{2x} = e^{2x} \quad \checkmark$$

$$y(0) = \frac{1}{2}e^{2\cdot 0} + \frac{1}{2} = \frac{1}{2}\underbrace{e^{0}}_{1} + \frac{1}{2} = 1 \quad \checkmark$$

We have proved the initial value problem was solved correctly. ☺

Applications to kinematics

We know that the **position** $s(t)$, **speed** $v(t)$ and **acceleration** $a(t)$
of a moving object are related by the following equations:

$v(t) = s'(t)$
$a(t) = v'(t) = s''(t)$.

This means that $s(t) = \int v(t)dt$ and $v(t) = \int a(t)dt$.

Problem 1 (Falling under gravity).
An object falling freely near the surface of the earth
is subject to a constant downward acceleration g (if air resistance is neglected).
If the initial height and velocity are y_0 and v_0, find the height $y(t)$ of the object at time t.

Solution. Given: $y(0) = y_0$, $v(0) = v_0$, $a(t) = g$.

To find: $y(t)$.

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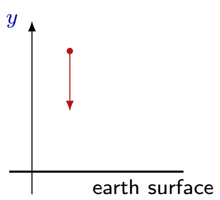
Falling under gravity



Let $y(t)$ be the height of the object
above the ground at time t.
Since the acceleration g is directed **down**,
$y''(t) = -g$.

We are given that $y(0) = y_0$, $y'(0) = v(0) = v_0$.

So the problem is reduced to finding a solution for the **initial value problem**

$$\begin{cases} y''(t) = -g \\ y(0) = y_0 \\ y'(0) = v_0. \end{cases}$$

Keep in mind that g, y_0, v_0 are given constants and $y = y(t)$ is an unknown function to be found.

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Falling under gravity

We start with integrating the differential equation $y''(t) = -g$:

$\int y''(t)dt = \int -g\,dt$ which gives $y'(t) = -gt + C_1$.

Since $y'(0) = v_0$, we find $v_0 = y'(0) = -g \cdot 0 + C_1 \implies C_1 = v_0$.

Therefore $y'(t) = -gt + v_0$. Integrate one more time:

$\int y'(t)dt = \int (-gt + v_0)dt$, which gives $y(t) = -\dfrac{gt^2}{2} + v_0 t + C_2$.

Since $y(0) = y_0$, we find $y_0 = y(0) = -\dfrac{g \cdot 0^2}{2} + v_0 \cdot 0 + C_2 \implies C_2 = y_0$.

Therefore, $\boxed{y(t) = -\dfrac{gt^2}{2} + v_0 t + y_0}$

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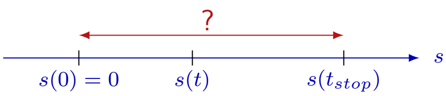
Stopping distance

Problem. A car is traveling at 60 mi/h when the brakes are fully applied, producing a constant deceleration of 4 ft/s^2. How far does the car travel before coming to a stop?

Solution.
Let $s(t)$ be the distance the car travels in t seconds after the braking.



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Stopping distance

We first calculate t_{stop}, the time it takes the car to stop.

Since $s''(t) = a(t) = -4$ ft/s^2, then the velocity is

$v(t) = s'(t) = \int s''(t)dt = \int -4dt = -4t + C_1$ ft/s.

Since $v(0) = 60$ mi/h $= 60 \cdot 5280/3600 = 88$ ft/s,

we have $88 = -4 \cdot 0 + C_1 \implies C_1 = 88.$

We have found that $s'(t) = v(t) = 88 - 4t$. Therefore,

$s(t) = \int s'(t)dt = \int (88 - 4t)dt = 88t - 2t^2 + C_2$.

Since $s(0) = 0$, we find $C_2 = 0$ and $s(t) = 88t - 2t^2$.

When the car has stopped, the velocity is 0, so

$0 = v(t_{stop}) = s'(t_{stop}) = 88 - 4t_{stop} \implies t_{stop} = 22$ s.

The stopping distance is then $s(22) = 88 \cdot 22 - 2 \cdot (22)^2 = 968$ ft.

☞ Drive safely!

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Summary

In this lecture we learned

- how to integrate simple functions using the table of antiderivatives
- how to find the **general solution** of some simple differential equations
- how to solve an **initial value problem** for a differential equation
- how to derive the equation for falling under gravity.

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Comprehension checkpoint

- Calculate the following integrals:

$$\int \left(5x^2 - \frac{x+2}{x^3}\right) dx, \quad \int \frac{1}{1+x^2}\, dx, \quad \int \frac{1}{\sqrt{1-x^2}}\, dx, \quad \int 2^x\, dx, \quad \int \frac{3}{x}\, dx.$$

- Solve the initial value problem

$$\begin{cases} y'' = x^2 - 2x + 3 \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$$

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