Lecture 25

Antiderivative and Indefinite Integral

Objectives

The rest of our course is devoted to **integration**.

We will discuss **indefinite and definite integrals** and elementary methods of integration.

In this lecture, we define an **antiderivative** of a function and study its properties.

We will define the indefinite integral of a function as its general antiderivative.

Differentiation vs integration

Differentiation and integration are the two main operations on functions in calculus. They are opposite in many ways.

Differentiation is easy to perform, but difficult to define.
Differential calculus appeared relatively recently, about 350 years ago.
The definition of derivative is not intuitive! But differentiation follows some simple rules, which make it easy to apply to elementary functions.
The derivative of an elementary function is an elementary function.
It is easy to compute, so differentiation is a routine operation.

Integration is difficult to perform, but easy to define.
The integral appeared in its simplest form, as an area, about 2500 years ago in Ancient Greece.
The definition of the integral, as opposed to the definition of the derivative, is simple and intuitive.But integrals of many elementary function can not be expressed in terms of elementary functions. And even when they can, integration is an art, not a routine!

Antiderivatives

Definition. An *antiderivative* of the function $f(x)$
is a differentiable function $F(x)$ such that $F'(x) = f(x)$.

Example 1. $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x) = x^2$, since

$$F'(x) = \frac{d}{dx}\left(\frac{1}{3}x^3\right) = \frac{1}{3} \cdot 3x^2 = x^2 = f(x).$$

Example 2. $F(x) = \frac{1}{3}x^3 + 5$ is also an antiderivative of $f(x) = x^2$, since

$$F'(x) = \frac{d}{dx}\left(\frac{1}{3}x^3 + 5\right) = \frac{1}{3} \cdot 3x^2 = x^2 = f(x).$$

Example 3. $F(x) = \sin x$ is an aniderivative of $f(x) = \cos x$, since

$$F'(x) = \frac{d}{dx}\sin x = \cos x = f(x).$$

⚠**Warning:**
Do not confuse a function f, its derivative f' and an antiderivative F:

$$\underset{\text{aniderivative}}{F(x)} \xrightarrow{\frac{d}{dx}} \underset{\text{function}}{f(x)} \xrightarrow{\frac{d}{dx}} \underset{\text{derivative}}{f'(x)}$$

The indefinite integral

☞ An aniderivative of a function is **not unique**:

if $F(x)$ is an antiderivative of $f(x)$, then $F(x) + C$, where C is an arbitrary constant,
is also an antiderivative, since $(F(x) + C)' = F'(x) = f(x)$.

Theorem. If $F(x)$ is an antiderivative of $f(x)$ on an interval I,
then any antiderivative of f on I is $F(x) + C$, where C is constant.

Proof. By a corollary of the Mean Value Theorem (see Lecture 18),
if $f'(x) = g'(x)$ for all x on some interval, then $g = f + C$.

So if $F(x)$ and $G(x)$ are two aniderivatives of $f(x)$, that is if
$F'(x) = f(x) = G'(x)$, then $G(x) = F(x) + C$.

Definition. The *indefinite integral* (or *general antiderivative*)
of $f(x)$ on an interval I is

$$\int f(x)dx = F(x) + C, \text{ where } F'(x) = f(x) \text{ for all } x \in I.$$

The integral sign

The symbol $\int$ is called the *integral sign*.

The expression $\int f(x)dx$ should be understood as a single symbol, representing the general antiderivative of f with respect to x.

By the definition of indefinite integral,

$$\frac{d}{dx}\left(\int f(x)dx\right) = f(x),$$

which means that differentiation and indefinite integration are inverse operations:

$$f(x) \underset{\text{differentiate}}{\overset{\text{integrate}}{\rightleftarrows}} \int f(x)dx$$

Examples of antiderivatives

$\int x\,dx = \frac{x^2}{2} + C$, since $\frac{d}{dx}\left(\frac{x^2}{2} + C\right) = x$.

$\int x^a\,dx = \frac{x^{a+1}}{a+1} + C$, for any $a \neq -1$. Indeed, $\frac{d}{dx}\left(\frac{x^{a+1}}{a+1} + C\right) = x^a$.

$\int \frac{1}{x}\,dx = \ln|x| + C$, since $\frac{d}{dx}(\ln|x| + C) = \frac{1}{x}$.

More explicitly, $\frac{d}{dx}\ln|x| = \frac{d}{dx}\begin{cases}\ln x, & x > 0\\ \ln(-x), & x < 0\end{cases} = \begin{cases}\frac{1}{x}, & x > 0\\ \frac{-1}{-x}, & x < 0\end{cases} = \frac{1}{x}.$

$\int e^x\,dx = e^x + C$, since $\frac{d}{dx}(e^x + C) = e^x$.

$\int a^x\,dx = \frac{a^x}{\ln a} + C$, since $\frac{d}{dx}\left(\frac{a^x}{\ln a}\right) = a^x$. Here, $a > 0$ and $a \neq 1$.

Functions and their derivatives

A table of the derivatives

$f(x)$	$f'(x)$
C	0
x^a	ax^{a-1}
e^x	e^x
a^x	$a^x \ln a$
$\ln x$	$\frac{1}{x}$
$\log_a x$	$\frac{1}{x \ln a}$

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\frac{1}{\cos^2 x}$
$\cot x$	$-\frac{1}{\sin^2 x}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$

can be easily converted to a table of antiderivatives.

Functions and their antiderivatives

$f(x)$	$F(x)$
x^a	$\frac{x^{a+1}}{a+1}$
$\frac{1}{x}$	$\ln \lvert x \rvert$
e^x	e^x
a^x	$\frac{a^x}{\ln a}$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$

$f(x)$	$F(x)$
$\frac{1}{\cos^2 x}$	$\tan x$
$\frac{1}{\sin^2 x}$	$-\cot x$
$\frac{1}{1+x^2}$	$\arctan x$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x$
$-\frac{1}{\sqrt{1-x^2}}$	$\arccos x$

Remember: $\underset{\text{function}}{f(x)} \xleftarrow{\frac{d}{dx}} \underset{\text{antiderivative}}{F(x)}$

Linearity of the anitiderivative

Theorem. $\int (f(x)+g(x))dx = \int f(x)dx + \int g(x)dx$

$\int Kf(x)dx = K\int f(x)dx$, where K is a constant.

Proof. Let $F(x)$, $G(x)$ be antiderivatives of $f(x)$ and $g(x)$ respectively,
that is, $F'(x) = f(x)$ and $G'(x) = g(x)$.

Then $(F(x)+G(x))' = F'(x)+G'(x) = f(x)+g(x)$,
that is, $F(x)+G(x)$ is an antiderivative of $f(x)+g(x)$ and
$\int (f(x)+g(x))dx = F(x)+G(x)+C = \int f(x)dx + \int g(x)dx$.

Since $(KF(x))' = KF'(x) = Kf(x)$, then $KF(x)$ is an antiderivative for $Kf(x)$ and
$\int Kf(x)dx = K\int f(x)dx$. □

Summary

In this lecture we learned

- what an antiderivative of a function is
(it is a function whose derivative is the original function)

- that an antiderivative of a function is not unique

- what an indefinite integral of a function is (it is the general antiderivative)

- properties of the antiderivative (linearity)

Comprehension checkpoint

- Fill in the table. Remember: $F(x) \xrightarrow{\frac{d}{dx}} f(x) \xrightarrow{\frac{d}{dx}} f'(x)$

an antiderivative $F(x)$	the function $f(x)$	its derivative $f'(x)$
	x	
x		
		x
	$\cos x$	
		$\sin x$
$5x^3 + 2x - 4$		
		$3x^2 + x + 1$
	e^x	
		2^x
3^x		
$\ln \lvert x \rvert$		

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