Lecture 24

Optimization Problems

Objectives

This is our final lecture devoted to applications of differentiation.

Today we discuss a special type of problems, called **optimization problems**,
or **extreme value problems**.

Optimization problems come from practical application of calculus.

They appear in all areas of human activities: natural and social sciences, engineering, business.

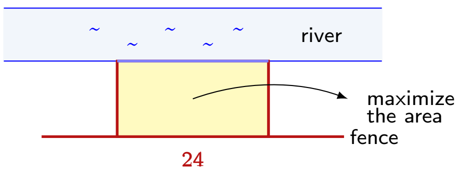
Mathematically, these problems are about finding a maximum or minimum value
of a function over a given interval.

Solutions for such problems are based on standard tools for analysis of functions:
limits and derivatives.

Fence by the river: problem

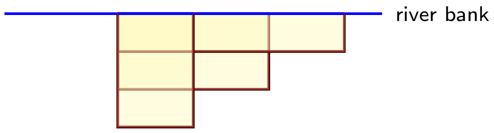
Problem. A rectangular enclosure bordering
a straight river has to be constructed.
If 24 yards of fence are available,
and no fence is needed along the river,
what is the largest possible area for the enclosure?

Discussion. Do we understand what's going on?



Fence by the river: discussion

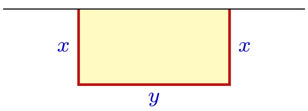
There are many ways to fence off a rectangle out of the given amount of fencing:



Which way gives the **largest** rectangle?
Why should a rectangle of the largest area exist at all?

To answer these questions, let us formulate the problem mathematically:

Problem. Maximize the area of a rectangle given the sum of its three sides (24 yards).



Solution. Introduce notations:
let x and y be the lengths of the sides of the rectangle.
Then the area of the rectangle is $A = xy$ (to maximize).
The length of the fence is $2x + y$, which is given to be 24 yards.
By this, the **constraint equation** is $2x + y = 24$.

Fence by the river: function and constraint

We have got a function to maximize, $A = xy$, and a constraint equation, $2x + y = 24$.
As is, the problem belongs to multivariable calculus
(and may be solved by methods of multivariable calculus).

We reduce it to a problem with **one** variable.

From the constraint equation we solve y in terms of x: $2x + y = 24 \implies y = 24 - 2x$ and substitute this into the function to maximize:

$A = xy = x(24 - 2x) = -2x^2 + 24x$.

So we have to maximize the function $A(x) = -2x^2 + 24x$. Over which interval? In other words, what is the domain for x?

The sides of our rectangle should have **positive** lengths, therefore, $x > 0$ and $y > 0$.
Since $y = 24 - 2x$, we get $24 - 2x > 0 \iff x < 12$.

Overall, $0 < x < 12$, that is $x \in (0, 12)$. This is the domain for x.

Fence by the river: solution

Here is problem: Maximize $A(x) = -2x^2 + 24x$ if $x \in (0, 12)$,
that is, find the maximal value of $A(x) = -2x^2 + 24x$ over the interval $(0, 12)$.

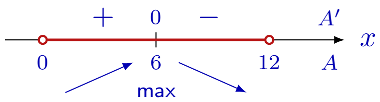
The function $A(x)$ is **continuous**, but the interval $(0, 12)$ is **not** closed.
So we can't guarantee that a solution exists.

Nevertheless, let us investigate the function over the interval.

Critical points: $A'(x) = 0 \iff -4x + 24 = 0 \iff x = 6$.

The only critical point is $x = 6$ and it belongs to our interval.
There are no singular points of the function nor boundary points of the interval.

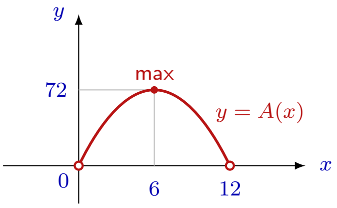
So $x = 6$ is the only nominee for the extreme point.
Is it a maximum or minimum? There are two ways to answer. Choose one.
Alternative 1. Sign study for A':



Fence by the river: completion

Alternative 2. Second derivative test. $A''(x) = (-4x + 24)' = -4$.
Therefore, $A''(6) = -4 < 0$ and $x = 6$ is a local (and global) maximum.

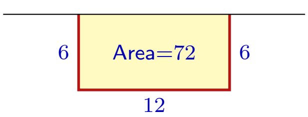
We may illustrate our calculation by graphing $A(x)$. Since $A(x) = -2x^2 + 24x = 2x(-x + 12)$, the parabola $y = A(x)$ looks as follows:



$A(6) = 2 \cdot 6 \cdot (-6 + 12) = 72.$

Therefore, $\max\limits_{(0,12)} A(x) = 72$ attained at $x = 6$.

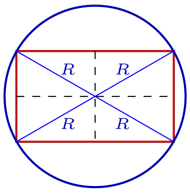
Answer. The largest possible enclosed area is 72 square yards.



Inscribed rectangle of largest area

Problem. What is the largest area of a rectangle
which can be inscribed in a circle of radius R?

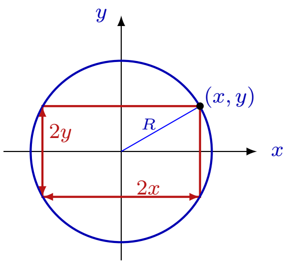
Solution. The center of a circle circumscribed around a rectangle is located at the intersection of the perpendicular bisectors of the rectangle's sides:



Therefore, the center of the rectangle coincides with the center of the circle.

Let us introduce a coordinate system, variables and function to maximize.

Area function



Let (x, y) be the coordinates
of the upper right corner of the rectangle.
Then the area of the rectangle is $A = 2x \cdot 2y = 4xy$.
We have to maximize the area under the constraint
$x^2 + y^2 = R^2$.

Solve y from the constraint equation: $x^2 + y^2 = R^2 \implies y = \sqrt{R^2 - x^2}$.

The function to maximize is $A(x) = 4x\sqrt{R^2 - x^2}$, where $x \in (0, R)$.

For the sake of simplicity, let us maximize the function $f(x) = x\sqrt{R^2 - x^2}$.
f takes its extreme values at the same points as A does.

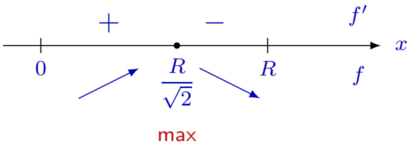
Do the math

We have to maximize $f(x) = x\sqrt{R^2 - x^2}$ on the interval $(0, R)$.

$$f'(x) = \sqrt{R^2 - x^2} + x \cdot (\sqrt{R^2 - x^2})' = \sqrt{R^2 - x^2} + x\frac{1}{2}\frac{-2x}{\sqrt{R^2 - x^2}}$$
$$= \sqrt{R^2 - x^2} - \frac{x^2}{\sqrt{R^2 - x^2}} = \frac{R^2 - 2x^2}{\sqrt{R^2 - x^2}}.$$

Critical points: $f'(x) = 0 \iff R^2 - 2x^2 = 0 \implies x = R/\sqrt{2}$, since $x > 0$.

There are no singular points on the interval $(0, R)$.
First derivative test (sign study for f'):

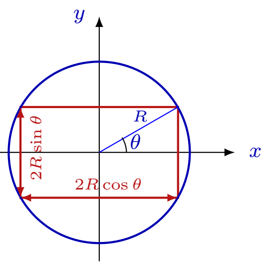


Therefore, $f(x) = x\sqrt{R^2 - x^2}$ attains its maximum at $x = R/\sqrt{2}$.

The maximum value of the area function is

$$A\left(\frac{R}{\sqrt{2}}\right) = 4f\left(\frac{R}{\sqrt{2}}\right) = 4\frac{R}{\sqrt{2}}\sqrt{R^2 - (R/\sqrt{2})^2} = 4\frac{R}{\sqrt{2}} \cdot \frac{R}{\sqrt{2}} = \boxed{2R^2}$$

Alternative solution



Let θ be the angle between the positive direction of x-axis and of the upper right corner of the rectangle. Then the area of the rectangle is
$A = (2R\cos\theta) \cdot (2R\sin\theta) = 2R^2\sin(2\theta)$.
We have to maximize
$A(\theta) = 2R^2\sin(2\theta)$ for $\theta \in (0, \pi/2)$.

Let us maximize $f(\theta) = \sin(2\theta)$ on $(0, \pi/2)$.
$f'(\theta) = 2\cos(2\theta)$; $f'(\theta) = 0 \iff \cos(2\theta) = 0 \implies 2\theta = \pi/2 \implies \theta = \pi/4$.

Actually, there is no need to take the derivative, since we know that $\sin(2\theta)$ has maximum of 1 attained at $\theta = \pi/4$.

Therefore, the area function $A(\theta)$ attains its maximum of $\boxed{2R^2}$ when $\theta = \pi/4$.

By this, the rectangle of the maximal area inscribed in circle is a **square**.

Optimal shape of a cylindrical can

Problem. Find the most economical shape of a cylindrical tin can.

Discussion. The problem is stated vaguely.
What does it mean "most economical"?
What does it mean "shape"?

"Most economic" can mean two different things:
- minimal total surface area for the given volume
- maximal volume for the given total surface area.

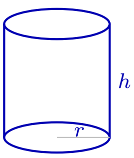
"Shape" of a cylinder is determined by its radius and height.

If we accept that "most economical" means having maximal volume for the given total surface area, then the problem takes the following form:

Problem. Among all cylinders of a given volume, find one with the minimal surface area.

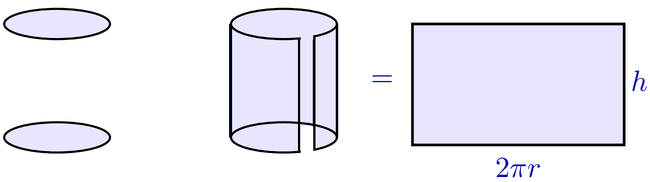
Volume and surface area

Solution. Draw a picture and introduce notations:



Let r be the radius of the cylinder and h be its height.

The volume of the cylinder is $V = \pi r^2 h$.
The total surface of the cylinder consists of two disks (top and bottom) and the lateral surface:



Therefore, the total surface area is $A = 2\pi r^2 + 2\pi r h$.

Function and constraint

We have to minimize $A = 2\pi r^2 + 2\pi rh$ under the constraint $V = \pi r^2 h$.

Keep in mind that V is a constant and A is a function in two variables, r and h.

In order to get a function in **one** variable, we eliminate h from the constraint:

$V = \pi r^2 h \implies h = \dfrac{V}{\pi r^2}$ and plug it into the expression for A:

$$A = 2\pi r^2 + 2\pi rh = 2\pi r^2 + 2\pi r \cdot \frac{V}{\pi r^2} = 2\pi r^2 + \frac{2V}{r}.$$

This gives us a function in one variable $A(r) = 2\pi r^2 + \dfrac{2V}{r}$.

What is the domain for the variable r?
The radius of the cylinder may be any positive number, that is, $r \in (0, \infty)$.

Our task is to find the minimum of $A(r) = 2\pi r^2 + \dfrac{2V}{r}$ if $r \in (0, \infty)$.

Finding the minimum

The function $A(r) = 2\pi r^2 + \dfrac{2V}{r}$ is differentiable on $(0, \infty)$,
so the extreme value may occur only at a critical point.

$A'(r) = 4\pi r - \dfrac{2V}{r^2}$, so $A'(r) = 0 \iff 4\pi r = \dfrac{2V}{r^2} \iff r^3 = \dfrac{V}{2\pi}$.

The only critical point of $A(r)$ is $r = \sqrt[3]{\dfrac{V}{2\pi}}$. Is it a maximum or minimum?

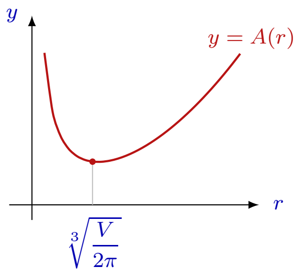
We can proceed with either a sign study for A' or the second derivative test.

But there is a more elegant way to resolve the situation.

$$A(r) = 2\pi r^2 + \frac{2V}{r} \xrightarrow[r \to 0^+]{} \infty, \quad A(r) = 2\pi r^2 + \frac{2V}{r} \xrightarrow[r \to \infty]{} \infty.$$

Therefore, the critical point should be a **minimum**.

The problem is solved!



$$\lim_{r\to 0^+} A(r) = \infty\,, \quad \lim_{r\to\infty} A(r) = \infty$$

We have found $r = \sqrt[3]{\dfrac{V}{2\pi}}$ where the minimum of $A(r)$ is attained.

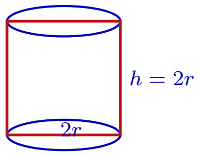
What is the shape of the cylinder then? That is h?

We know that $V = \pi r^2 h$ and $r^3 = \dfrac{V}{2\pi}$ for the critical point. Therefore,

$\pi r^2 h = V = 2\pi r^3 \implies \boxed{h = 2r}$ the height is equal to the diameter of the cylinder.

Optimal shape of a cylindrical can: the answer

We have got that
among all cylinders with a given volume, the cylinder whose height equals to the diameter has the minimal total surface area.



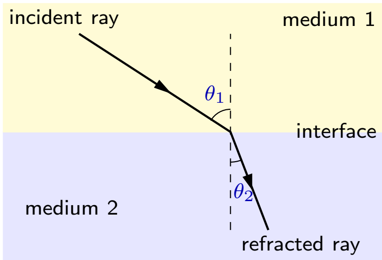
The vertical cross section of this cylinder is a square.

Extra problem. Among all cylinders with a given total surface area, find a cylinder with the maximal possible volume.

Spoiler: the answer remains the same!

The law of refraction

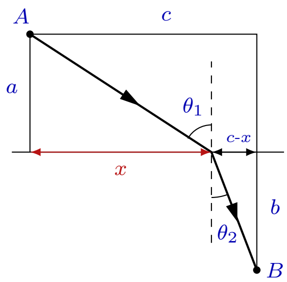
Snell's law (the law of refraction).
If light travels with speed v_1 in one medium and speed v_2 in the second medium, and if the media are separated by a plane interface,
then the ratio of the sines of the angles of incidence and refraction
is equal to the ratio of velocities in the two media.



$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

Proof of the law of refraction

According to Fermat's principle, the light travels the path which takes the least time.
Draw a picture and introduce notations:



The time required for the light to travel from A to B is

$$T(x) = \frac{\sqrt{x^2+a^2}}{v_1} + \frac{\sqrt{(c-x)^2+b^2}}{v_2}$$

To minimize T, we find the critical points:

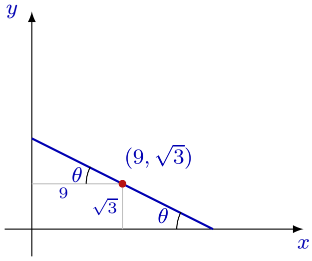
$$0 = T'(x) = \frac{1}{v_1}\frac{x}{\sqrt{x^2+a^2}} - \frac{1}{v_2}\frac{c-x}{\sqrt{(c-x)^2+b^2}} = \frac{1}{v_1}\sin\theta_1 - \frac{1}{v_2}\sin\theta_2\,.$$

Therefore, $\dfrac{1}{v_1}\sin\theta_1 - \dfrac{1}{v_2}\sin\theta_2 = 0 \iff \dfrac{\sin\theta_1}{\sin\theta_2} = \dfrac{v_1}{v_2}$, as required.

Shortest segment

Problem. What is the length of the shortest line segment having one end on the x-axis, the other end in the y-axis, and passing through the point $(9, \sqrt{3})$.

Solution.



Let θ be the angle shown.

Then the length of the line segment is

$$L(\theta) = \frac{9}{\cos\theta} + \frac{\sqrt{3}}{\sin\theta}, \quad \text{where } 0 < \theta < \pi/2.$$

Observe that $L \to \infty$ as $\theta \to 0^+$ or $\theta \to \frac{\pi}{2}^-$.

Therefore, $L(\theta)$ has the global **minimum** on $(0, \pi/2)$.

Finding the minimum

$$L'(\theta) = \left(\frac{9}{\cos\theta} + \frac{\sqrt{3}}{\sin\theta}\right)' = \frac{9\sin\theta}{\cos^2\theta} - \frac{\sqrt{3}\cos\theta}{\sin^2\theta}.$$

For critical points, $L'(\theta) = 0 \iff \dfrac{9\sin^3\theta - \sqrt{3}\cos^3\theta}{\cos^2\theta\sin^2\theta} = 0$

$$\iff \tan^3\theta = \frac{\sqrt{3}}{9} \iff \tan^3\theta = \left(\frac{1}{\sqrt{3}}\right)^3 \iff \theta = \frac{\pi}{6}.$$

The length of the shortest line segment is

$$L\left(\frac{\pi}{6}\right) = \frac{9}{\cos\frac{\pi}{6}} + \frac{\sqrt{3}}{\sin\frac{\pi}{6}} = \frac{9}{\sqrt{3}/2} + \frac{\sqrt{3}}{1/2} = 6\sqrt{3} + 2\sqrt{3} = \boxed{8\sqrt{3}}$$

Answer. The length of the shortest line segment is $8\sqrt{3}$.

Summary

In this lecture we learned how to solve **optimization problems**
by composing a function and finding its **extreme values**.

If necessary, revise the content of Lectures 17 and 19
to refresh your memories about finding extema of a function.

Optimization problems provide various opportunities in choosing a variable.
A good choice of the variable may simplify your solution.