Lecture 23

Related Rates

Objectives

This lecture is devoted to one of applications of differentiation.

We will discuss a special type of problem, called **related rates**,

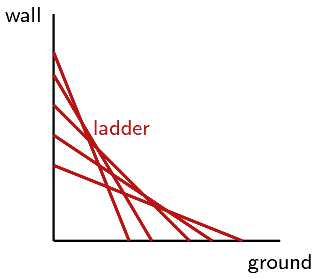
where the rates of change of related quantities are analyzed.

Sliding ladder: problem

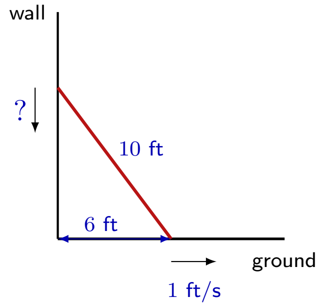
Problem. A ladder 10 ft long rests against a vertical wall.
If the bottom of the ladder slides away from the wall at a rate of 1 ft/s,
how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

Discussion. Do we understand what's going on?

Draw a picture:



Sliding ladder: discussion



Which numbers are **given**?
10 ft is the **length** of ladder,
1 ft/s is the **rate**
at which the bottom moves along the ground,
6 ft is the **distance** from the bottom to the wall.

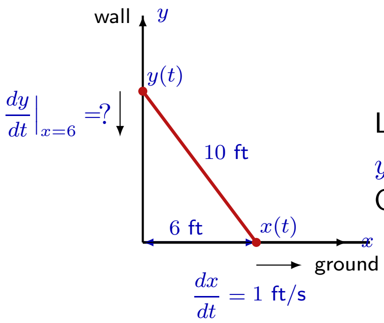
What do we have to **find**? The **rate** at which the top is sliding down.

Important: the situation changes with time. The rates and distances change, the only quantity which stays unchanged is the length of the ladder.

Now we understand what's going on and can start to solve the problem.

Sliding ladder: solution

Solution. Step 1. Draw the picture and introduce notations:



Let $x(t)$ be the position of the bottom at time t,
$y(t)$ be the position of the top at time t.
Observe that the positions depend on t.

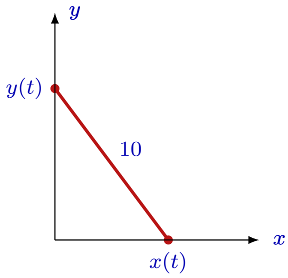
Step 2. Express what is given and what is to find in terms of introduced notations.

The bottom moves to the right at rate 1 ft/s, which means that $\frac{dx}{dt} = 1$ ft/s.
We have to find the rate at which the top slides down
at the moment when $x = 6$, that is, $\frac{dy}{dt}\Big|_{x=6} = ?$

Sliding ladder: constraint equation

Step 3. Write down the **constraint equation**, that is the equation connecting the quantities involved.

How are $x(t)$ and $y(t)$ related?



By the Pythagorean theorem,

$$\boxed{x^2(t) + y^2(t) = 10^2}$$

This is the **constraint equation** for the problem.

Sliding ladder: implicit differentiation

Step 4. Differentiate the constraint equation implicitly with respect to t:

$x^2(t) + y^2(t) = 10^2 \xrightarrow{\frac{d}{dt}} 2x(t)\dfrac{dx}{dt} + 2y(t)\dfrac{dy}{dt} = 0$, or, equivalently,

$x(t)\dfrac{dx}{dt} + y(t)\dfrac{dy}{dt} = 0 \quad (*)$. Remember that we are looking for $\left.\dfrac{dy}{dt}\right|_{x=6}$.

When $x = 6$, then $y = 8$ (since $x^2 + y^2 = 10^2$).

We are given that $\dfrac{dx}{dt} = 1$. Substitute these numbers into $(*)$:

$$6 \cdot 1 + 8\left.\frac{dy}{dt}\right|_{x=6} = 0 \implies \left.\frac{dy}{dt}\right|_{x=6} = \boxed{-\frac{3}{4}}$$

Notice that the negative value for dy/dt means that

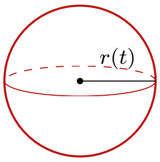
the distance from the top of the latter to the ground **decreases**.

Answer. The top of the ladder is sliding down at the rate of $\dfrac{3}{4}$ ft/s.

Inflating a balloon: problem

Problem. Air is pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

Solution. Draw a picture and introduce notations:



Let $r(t)$ be the radius on the balloon at time moment t and $V(t)$ be the volume on the balloon at time moment t.

What is **given**? The rate of change of the volume: $\frac{dV}{dt} = 100\,$cm^3/s.

What is to **find**? The rate of change of the radius: $\left.\frac{dr}{dt}\right|_{r=25} = ?$

Inflating a balloon: solution

How the quantities are related? $\boxed{V(t) = \frac{4}{3}\pi r^3(t)}$

This is the **constraint equation**.

Observe that both the radius $r(t)$ and volume $V(t)$ depend on time t.

Implicit differentiation with respect to t gives rise to

$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2(t)\frac{dr}{dt}$, or, equivalently, $\frac{dV}{dt} = 4\pi r^2(t)\frac{dr}{dt} \quad (*)$.

We are looking for $\left.\frac{dr}{dt}\right|_{r=25}$. Substitute $r = 25$ and $\frac{dV}{dt} = 100$ into $(*)$:

$100 = 4\pi(25)^2\left.\frac{dr}{dt}\right|_{r=25}$ and get $\left.\frac{dr}{dt}\right|_{r=25} = \boxed{\frac{1}{25\pi}}$

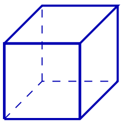
Check the units. Since $\frac{dr}{dt} = \frac{dV/dt}{4\pi r^2}$, we get $[dr/dt] = [dV/dt]/[r^2]$= $\frac{\text{cm}^3/\text{ s}}{\text{cm}^2}$ =cm/s.

Answer. The rate of radius increase is $\frac{1}{25\pi}$ cm/s.

Melting ice cube: beginning

Problem. An ice cube is melting.
At the moment when the volume of the cube is 8 in^3,
its surface area is decreasing at a rate of 2 in^2/min.
At what rate is the volume decreasing at this moment?

Solution. Draw a picture and introduce notations:



$x(t)$

Let $x(t)$ be the length of a side of the cube at time moment t.
Then at time t the volume is $V(t) = x^3(t)$
and the surface area is $A(t) = 6x^2(t)$.

Given: The rate of decreasing of the surface area $\left.\dfrac{dA}{dt}\right|_{V=8} = 2$ in^2/min.

Find: The rate of decreasing of the volume at this moment $\left.\dfrac{dV}{dt}\right|_{V=8} = ?$

Melting ice cube: completion

There are two **constraint equations**: $V(t) = x^3(t)$ and $A(t) = 6x^2(t)$.

Implicit differentiation with respect to t gives rise to

$$\frac{dV}{dt} = 3x^2(t)\frac{dx}{dt} \quad (*) \quad \text{and} \quad \frac{dA}{dt} = 12x(t)\frac{dx}{dt} \quad (**).$$

We are given $\left.\dfrac{dA}{dt}\right|_{V=8} = 2$ and have to find $\left.\dfrac{dV}{dt}\right|_{V=8}$.

From $(**)$ we get $\dfrac{dx}{dt} = \dfrac{dA/dt}{12x(t)}$, and substitute this into $(*)$:

$$\frac{dV}{dt} = 3x^2(t)\frac{dx}{dt} = 3x^2(t)\frac{dA/dt}{12x(t)} = \frac{1}{4}x(t)\frac{dA}{dt}.$$

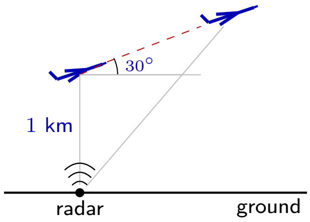
When $V = 8$, then $x = 2$ (since $V = x^3$) and $\left.\dfrac{dV}{dt}\right|_{V=8} = \dfrac{1}{4} \cdot 2 \cdot 2 = 1$ in^3/min.

Answer. The volume of the cube is decreasing at the rate of 1 in^3/min.

Flying airplane: problem

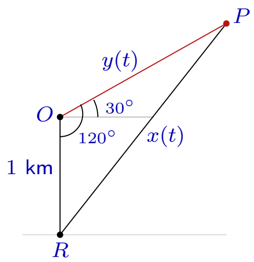
Problem. A plane flying at a constant speed of 300 km/h passes over a ground radar station at an altitude of 1 km and climbs at an angle of $30°$.
At what rate is the distance from the plane to the radar station increasing a minute later?

Discussion. Do we understand what's going on?



Flying airplane: solution

Solution. Draw a picture and introduce notations:



Let t be time (in hours) from the moment when the plane passed by the radar,
$x(t)$ be the distance (in km) from the plane to the radar,
$y(t)$ be the distance $|OP|$, in km.

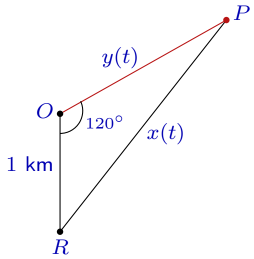
Given: $|RO| = 1$ km, $\angle ROP = 120°$, the speed of the plane $\frac{dy}{dt} = 300$ km/h.

Find: How is the distance from the plane to the radar increasing at $t = 1$ min,

that is, $\left.\frac{dx}{dt}\right|_{t=\frac{1}{60}} = ?$

Flying airplane: the constraint equation

What is the relation between the quantities involved?



By the **Cosine law**,

$|RP|^2 = |RO|^2 + |OP|^2 - 2|RO||OP| \cos \angle ROP$.

In our notations,

$x^2(t) = 1^2 + y^2(t) - 2 \cdot 1 \cdot y(t) \cos 120°$.

Since $\cos 120° = -1/2$, we get

$\boxed{x^2(t) = 1 + y^2(t) + y(t)}$ the **constraint equation** for the problem

Implicit differentiation with respect to t gives rise to

$2x(t)\dfrac{dx}{dt} = 2y(t)\dfrac{dy}{dt} + \dfrac{dy}{dt}$ $(*)$. We need to find $\left.\dfrac{dx}{dt}\right|_{t=\frac{1}{60}}$.

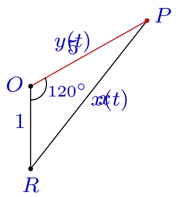
For this, solve $(*)$ for $\dfrac{dx}{dt}$.

Flying airplane: completion

$$2x(t)\frac{dx}{dt} = 2y(t)\frac{dy}{dt} + \frac{dy}{dt} \implies 2x(t)\frac{dx}{dt} = (2y(t)+1)\frac{dy}{dt} \implies \frac{dx}{dt} = \frac{2y(t)+1}{2x(t)}\frac{dy}{dt}.$$

Hence $\left.\dfrac{dx}{dt}\right|_{t=\frac{1}{60}} = \dfrac{2y(\frac{1}{60})+1}{2x(\frac{1}{60})} \left.\dfrac{dy}{dt}\right|_{t=\frac{1}{60}}$. What is the situation at $t = \dfrac{1}{60}$,

that is 1 min after the plane passed the radar?



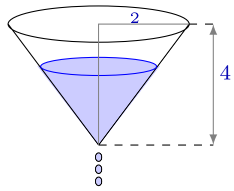
The plane is flying at a constant speed of 300 km/h.
Therefore, at $t = \frac{1}{60}$, it will be at the distance
$y(\frac{1}{60}) = 300 \cdot \frac{1}{60} = 5$ km from O.
The distance from the plane to the radar then is
$x^2 = 1 + 5^2 + 5 \implies x = \sqrt{31}$.

Therefore, $\left.\dfrac{dx}{dt}\right|_{t=\frac{1}{60}} = \dfrac{2y(\frac{1}{60})+1}{2x(\frac{1}{60})} \left.\dfrac{dy}{dt}\right|_{t=\frac{1}{60}} = \dfrac{2 \cdot 5 + 1}{2 \cdot \sqrt{31}} \cdot 300 = \dfrac{1650}{\sqrt{31}} \approx 296$ km/h

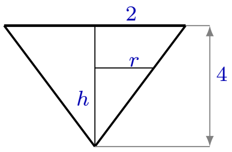
Answer: The distance from the plane to the radar is increasing at the rate of 296 km/h.

Leaking tank: problem

Problem. A leaky water tank has the shape of an inverted circular cone with height 4 m and base radius 2 m.
When the water in the tank is 3 m deep
it is leaking out at a rate of $1/12$ m^3/min.
How fast is the water level in the tank dropping at that time?



Solution. Draw a picture and introduce notation:



Let r and h be the radius and the depth of water at time moment t.
Then the volume of water at this time moment is
$V = \dfrac{1}{3}\pi r^2 h$.

Given: The rate of leaking $\left.\dfrac{dV}{dt}\right|_{h=3} = -\dfrac{1}{12}$.

Find: The rate at which the height of water decreases $\left.\dfrac{dh}{dt}\right|_{h=3} =?$

Leaking tank: constraint equations

The **constraint equations** describe how the quantities involved depend on each other:

$V = \dfrac{1}{3}\pi r^2 h$ and $\dfrac{r}{h} = \dfrac{2}{4}$ (from similar triangles).

We are looking for $\left.\dfrac{dh}{dt}\right|_{h=3}$. So it makes sense to express r in terms of h:

$\dfrac{r}{h} = \dfrac{2}{4} \implies r = h/2$ and substitute it into the equation for V:

$$V = \frac{1}{3}\pi r^2 h \implies V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h \iff V = \frac{\pi}{12}h^3, \text{ or } V(t) = \frac{\pi}{12}h^3(t)$$

Differentiate the latter equation implicitly with respect to t:

$\dfrac{dV}{dt} = \dfrac{\pi}{12}\cdot 3h^2(t)\dfrac{dh}{dt} \iff \dfrac{dV}{dt} = \dfrac{\pi}{4}h^2(t)\dfrac{dh}{dt}$. When $h = 3$, we get

$-\dfrac{1}{12} = \dfrac{\pi}{4}\cdot 3^2 \left.\dfrac{dh}{dt}\right|_{h=3} \implies \left.\dfrac{dh}{dt}\right|_{h=3} = -\dfrac{1}{27\pi} \approx -0.012$ m/min.

Answer: The water level drops at a rate 0.012 m/min.

Summary

In this lecture we learned how to solve **related rates** problems.

Solutions involve composing **constraint equations** and **implicit differentiation**.

Drawing pictures always helps to solve these problems.