Lecture 22

Indeterminate Forms and L'Hôpital's rule

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Objectives

In this lecture, we will study **indeterminate forms** and their types.

We will learn **L'Hôpital's rule** which helps to calculate limits involving indeterminate forms.

We will apply L'Hôpital's rule for finding **asymptotes** of the graphs of functions.

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What is an indeterminate form?

In Lecture 14, we shown that $\lim_{x\to 0} \frac{\sin x}{x} = 1$. This limit can not be calculated by direct substitution of $x = 0$ into $\frac{\sin x}{x}$, since it results in $\frac{0}{0}$.

We call $\frac{\sin x}{x}$ an *indeterminate form* of type $\left[\frac{0}{0}\right]$.

The limit of such an indeterminate form can be **any** number, ∞, $-\infty$, or it does not exist at all.

For example, $\frac{2x}{x}$, $\frac{x^2}{x}$, $\frac{x}{x^3}$, $\frac{x}{x^2}$ are indeterminate forms of the same type $\left[\frac{0}{0}\right]$, but

$\lim_{x\to 0} \frac{2x}{x} = \lim_{x\to 0} 2 = 2$, $\quad \lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x = 0$,

$\lim_{x\to 0} \frac{x}{x^3} = \lim_{x\to 0} \frac{1}{x^2} = \infty$, $\quad \lim_{x\to 0} \frac{x}{x^2} = \lim_{x\to 0} \frac{1}{x}$ does not exist.

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Types of indeterminate forms

Type	Example
$[0/0]$	$\lim_{x\to 0} \frac{\sin x}{x}$
$[\infty/\infty]$	$\lim_{x\to\infty} \frac{x}{e^x}$
$[0\cdot\infty]$	$\lim_{x\to 0^+} x \ln \frac{1}{x}$
$[\infty-\infty]$	$\lim_{x\to\infty} (\sqrt{x^2+x}-x)$
$[0^0]$	$\lim_{x\to 0^+} x^x$
$[\infty^0]$	$\lim_{x\to\infty} x^{1/x}$
$[1^\infty]$	$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$

Methods of evaluating indeterminate forms

- Algebraic transformations

$$\lim_{x\to 1} \frac{x^2-1}{x-1} \quad \left[\frac{0}{0}\right] = \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1}(x+1) = 2$$

$$\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-2x}}{x} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{(\sqrt{1+x}-\sqrt{1-2x})(\sqrt{1+x}+\sqrt{1-2x})}{x(\sqrt{1+x}+\sqrt{1-2x})} = \lim_{x\to 0} \frac{(1+x)-(1-2x)}{x(\sqrt{1+x}+\sqrt{1-2x})}$$

$$= \lim_{x\to 0} \frac{3x}{x(\sqrt{1+x}+\sqrt{1-2x})} = \lim_{x\to 0} \frac{3}{\sqrt{1+x}+\sqrt{1-2x}} = \frac{3}{2}$$

- Taylor's expansion (to be studied in MAT 132)
- L'Hôpital's rule

L'Hôpital's rule

Theorem. Let f, g be differentiable functions and $g'(x) \neq 0$ near a (except possibly at a) and $\dfrac{f(x)}{g(x)}$ is an **indeterminate form** of type $\left[\dfrac{0}{0}\right]$ or $\left[\dfrac{\infty}{\infty}\right]$.

Then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$, if the latter limit exists.

Proof will be given in the course of Analysis.

☞ Be careful using L'Hôpital's rule:

- check first that the indeterminate form is indeed $[0/0]$ or $[\infty/\infty]$
- take the derivatives of the numerator and denominator **separately**, not the derivative of the quotient
- the rule is applicable only in the limit of the quotient of the derivatives **exists**.

$[0/0]$

Example 1. $\lim\limits_{x\to 0} \dfrac{\sin x}{x} \quad \left[\dfrac{0}{0}\right] = \lim\limits_{x\to 0} \dfrac{\dfrac{d}{dx}\sin x}{\dfrac{d}{dx}x} = \lim\limits_{x\to 0} \dfrac{\cos x}{1} = \dfrac{\cos 0}{1} = 1.$

Example 2. $\lim\limits_{x\to 1} \dfrac{\ln x}{x-1} \quad \left[\dfrac{\ln 1}{1-1} = \dfrac{0}{0}\right] = \lim\limits_{x\to 1} \dfrac{\dfrac{d}{dx}\ln x}{\dfrac{d}{dx}(x-1)} = \lim\limits_{x\to 1} \dfrac{1/x}{1} = 1.$

Example 3. $\lim\limits_{x\to\infty} \dfrac{\tan\dfrac{1}{x}}{\arctan x - \pi/2} \quad \left[\dfrac{\tan 0}{\pi/2 - \pi/2} = \dfrac{0}{0}\right]$

$$= \lim_{x\to\infty} \frac{\left(\tan\dfrac{1}{x}\right)'}{(\arctan x - \pi/2)'} = \lim_{x\to\infty} \frac{\dfrac{1}{\cos^2(1/x)}\left(-\dfrac{1}{x^2}\right)}{\dfrac{1}{1+x^2}} = \lim_{x\to\infty} \left(-\frac{1+x^2}{x^2}\right)\frac{1}{\cos^2(1/x)}$$

$$= \lim_{x\to\infty} \left(-\frac{1}{x^2} - 1\right)\frac{1}{\cos^2(1/x)} = (0-1)\frac{1}{\cos^2 0} = -1\,.$$

$[\infty/\infty]$

Example 1. $\lim\limits_{x\to\infty} \dfrac{x^2}{e^x} \left[\dfrac{\infty}{\infty}\right] = \lim\limits_{x\to\infty} \dfrac{(x^2)'}{(e^x)'} = \lim\limits_{x\to\infty} \dfrac{2x}{e^x} \left[\dfrac{\infty}{\infty}\right]$

$$= \lim_{x\to\infty} \frac{(2x)'}{(e^x)'} = \lim_{x\to\infty} \frac{2}{e^x} = 0.$$

Remark. This example shows that the exponential function grows **faster** than the square function.

Let n be a positive integer and a be a real number greater than 1. Then

$$\lim_{x\to\infty} \frac{x^n}{a^x} \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty} \frac{(x^n)'}{(a^x)'} = \lim_{x\to\infty} \frac{nx^{n-1}}{a^x \ln a} \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty} \frac{(nx^{n-1})'}{(a^x \ln a)'} =$$

$\lim\limits_{x\to\infty} \dfrac{n(n-1)x^{n-2}}{a^x(\ln a)^2} \left[\dfrac{\infty}{\infty}\right] = \cdots = \lim\limits_{x\to\infty} \dfrac{n!}{a^x(\ln a)^n} = 0,$ since $n!$ and $(\ln a)^n$ are both constants.

☞ Any exponential with base greater than 1 grows **faster** than any power with exponent greater than 0.

$[\infty/\infty]$

Example 2. Which function does grow faster, logarithmic or power?

Let $a > 0$. Then

$$\lim_{x\to\infty} \frac{\ln x}{x^a} \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty} \frac{(\ln x)'}{(x^a)'} = \lim_{x\to\infty} \frac{1/x}{ax^{a-1}} = \lim_{x\to\infty} \frac{1}{ax^a} = 0.$$

☞ Any power with a positive exponent grows **faster** than logarithm.

Example 3. $\lim\limits_{x\to-\infty} \dfrac{x}{e^{-x}} \left[\dfrac{-\infty}{\infty}\right] = \lim\limits_{x\to-\infty} \dfrac{x'}{(e^{-x})'} = \lim\limits_{x\to-\infty} \dfrac{1}{-e^{-x}} = 0.$

Example 4. $\lim\limits_{x\to0^+} (\sin x)(\ln x) = \lim\limits_{x\to0^+} \dfrac{\ln x}{1/\sin x} \left[\dfrac{-\infty}{\infty}\right] = \lim\limits_{x\to0^+} \dfrac{1/x}{-(\cos x)/(\sin^2 x)}$

$$= -\lim_{x\to0^+} \frac{\frac{\sin^2 x}{x}}{\cos x} = -\lim_{x\to0^+} \frac{\sin^2 x}{x^2} \cdot \frac{x}{\cos x} = -\lim_{x\to0^+} \left(\frac{\sin x}{x}\right)^2 \cdot \frac{x}{\cos x} = -1^2 \cdot \frac{0}{1} = 0\,.$$

$[0\cdot\infty]$ and $[\infty-\infty]$

Example 1. $\lim\limits_{x\to0+} x\ln\frac{1}{x}\quad [0\cdot\infty] = \lim\limits_{x\to0+}\frac{\ln\frac{1}{x}}{\frac{1}{x}}\quad \left[\frac{\infty}{\infty}\right] = \lim\limits_{x\to0+}\frac{(\ln\frac{1}{x})'}{(\frac{1}{x})'}$

$= \lim\limits_{x\to0+}\frac{x\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim\limits_{x\to0+} x = 0.$

Remark. One can calculate $\lim\limits_{x\to0+}\frac{\ln\frac{1}{x}}{\frac{1}{x}}$ more efficiently:

$$\lim_{x\to0+}\frac{\ln\frac{1}{x}}{\frac{1}{x}} = \lim_{t\to\infty}\frac{\ln t}{t} = \lim_{t\to\infty}\frac{(\ln t)'}{t'} = \lim_{t\to\infty}\frac{1/t}{1} = 0.$$

Example 2. $\lim\limits_{x\to\infty}(x-\ln x)\quad [\infty-\infty] = \lim\limits_{x\to\infty} x\left(1-\frac{\ln x}{x}\right).$

We know that $\lim\limits_{x\to\infty}\frac{\ln x}{x}\quad \left[\frac{\infty}{\infty}\right] = \lim\limits_{x\to\infty}\frac{(\ln x)'}{x'} = \lim\limits_{x\to\infty}\frac{1/x}{1} = 0\,.$

Therefore $\lim\limits_{x\to\infty} x\left(1-\frac{\ln x}{x}\right) = \infty.$

$[0^0]$, $[\infty^0]$ and $[1^\infty]$

☞ The easiest way to deal with $[0^0]$, $[\infty^0]$ and $[1^\infty]$ is **to take logarithms** of the expressions involved.

Example 1. $\lim\limits_{x\to0+} x^x\quad [0^0] = \lim\limits_{x\to0+} e^{\ln(x^x)} = \lim\limits_{x\to0+} e^{x\ln x}.$

Calculate $\lim\limits_{x\to0+} x\ln x\,$:

$$\lim_{x\to0+} x\ln x\quad [0\cdot(-\infty)] = \lim_{x\to0+}\frac{\ln x}{\frac{1}{x}}\left[\frac{-\infty}{\infty}\right] = \lim_{x\to0+}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to0+}(-x) = 0\,.$$

So $x\ln x \underset{x\to0+}{\longrightarrow} 0$ and therefore $e^{x\ln x}\underset{x\to0+}{\longrightarrow} e^0 = 1\,.$

Finally, $\lim\limits_{x\to0+} x^x = 1\,.$

$[\infty^0]$

Example 2. $\lim_{x\to\infty} x^{1/x}$ $[\infty^0] = \lim_{x\to\infty} e^{\ln(x^{1/x})} = \lim_{x\to\infty} e^{\frac{1}{x}\ln x}$.

Calculate $\lim_{x\to\infty} \frac{1}{x}\ln x$:

$$\lim_{x\to\infty} \frac{1}{x}\ln x = \lim_{x\to\infty} \frac{\ln x}{x} \quad \left[\frac{\infty}{\infty}\right] = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0.$$

So $\frac{1}{x}\ln x \underset{x\to\infty}{\longrightarrow} 0$ and therefore $e^{\frac{1}{x}\ln x} \underset{x\to\infty}{\longrightarrow} e^0 = 1$.

Finally, $\lim_{x\to\infty} x^{1/x} = 1$.

$[1^\infty]$

Example 3. $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$ $[1^\infty] = \lim_{x\to\infty} e^{\ln(1+\frac{1}{x})^x} = \lim_{x\to\infty} e^{x\ln\left(1+\frac{1}{x}\right)}$.

Calculate $\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)$:

$$\lim_{x\to\infty} x\ln\left(1+\tfrac{1}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \quad \left[\frac{0}{0}\right] = \lim_{t\to0^+} \frac{\ln(1+t)}{t} = \lim_{t\to0^+} \frac{\frac{1}{1+t}}{1} = 1.$$

So $x\ln\left(1+\frac{1}{x}\right) \underset{x\to\infty}{\longrightarrow} 1$ and therefore $e^{x\ln\left(1+\frac{1}{x}\right)} \underset{x\to\infty}{\longrightarrow} e^1 = e$.

Finally, $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$.

Warning

⚠ **Warning:** Before applying L'Hôpital's rule, always check whether the expression is indeed an indeterminate form.

Example. Find the limit $\lim_{x\to 0} \dfrac{x+1}{\sin x}$.

Incorrect: $\lim_{x\to 0} \dfrac{x+1}{\sin x} = \lim_{x\to 0} \dfrac{(x+1)'}{(\sin x)'} = \lim_{x\to 0} \dfrac{1}{\cos x} = \dfrac{1}{\cos 0} = 1$.

Correct: $\lim_{x\to 0} \dfrac{x+1}{\sin x} \left[\dfrac{1}{0}\right]$ which is **not** an indeterminate form.

Calculate right and left limits:

$$\lim_{x\to 0^+} \frac{x+1}{\sin x} = \left[\frac{1}{0^+}\right] = \infty, \quad \lim_{x\to 0^-} \frac{x+1}{\sin x} = \left[\frac{1}{0^-}\right] = -\infty.$$

Therefore $\lim_{x\to 0^+} \dfrac{x+1}{\sin x} \neq \lim_{x\to 0^-} \dfrac{x+1}{\sin x}$ and, by this, $\lim_{x\to 0} \dfrac{x+1}{\sin x}$ does not exist.

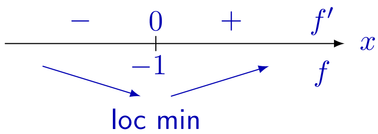
L'Hôpital's rule for finding asymptotes

Example. For the function $f(x) = xe^x$, find the asymptotes and draw the graph.

Solution. $f'(x) = (xe^x)' = e^x + xe^x = (1+x)e^x$.

Critical points: $f'(x) = 0 \iff (1+x)e^x = 0 \iff \boxed{x = -1}$ since $e^x > 0$ for all x.

Increasing/decreasing test (sign study for f'):



$f(-1) = -1e^{-1} = -\dfrac{1}{e}$ is the local (and absolute) minimum.

Zeros of the function (x-intercepts): $f(x) = 0 \iff xe^x = 0 \iff \boxed{x = 0}$.

L'Hôpital's rule for finding asymptotes

The function $f(x) = xe^x$ is defined for **all** x, so there are no vertical asymptotes.

For the horizontal asymptotes, calculate the limits $\lim_{x \to \pm\infty} f(x)$:

$\lim_{x \to \infty} xe^x \; [\infty \cdot \infty] = \infty$, so there is no asymptote as $x \to \infty$.

$$\lim_{x \to -\infty} xe^x \; [-\infty \cdot 0] = \lim_{x \to -\infty} \frac{x}{e^{-x}} \; \left[\frac{-\infty}{\infty}\right] = \lim_{x \to -\infty} \frac{x'}{(e^{-x})'} = \lim_{x \to -\infty} \frac{1}{-e^{-x}} = 0^- .$$

Hence $y = 0$ is the horizontal asymptote at $x \to -\infty$,

and the graph approaches this asymptote from **below**.

Now we assemble all the information obtained for drawing the graph.

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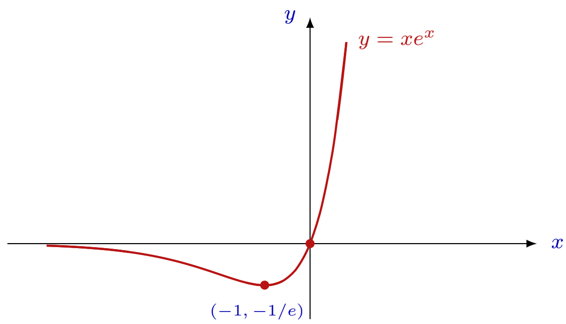
L'Hôpital's rule for finding asymptotes



Draw a small "cup" for the local minimum at $(-1, -1/e)$.
Indicate the x-intercept $(0, 0)$.
Draw a piece of the graph for $x \to -\infty$.
Draw a piece of the graph for $x \to \infty$.
Connect the constructed pieces by a smooth curve.

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Summary

In this lecture we learned how to calculate the limits of **indeterminate forms.**

Remember which expressions fall into which category of indeterminate forms:

$$\left[\frac{0}{0}\right], \quad \left[\frac{\infty}{\infty}\right], \quad [0\cdot\infty], \quad [\infty-\infty], \quad \left[0^0\right], \quad \left[\infty^0\right], \quad \left[1^\infty\right].$$

The limits of the expressions of types $\left[\frac{0}{0}\right]$ and $\left[\frac{\infty}{\infty}\right]$ are calculated

by **L'Hôpital's rule**: $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$

Other types of indeterminate forms are treated by studying the natural logarithms of the given forms.

Comprehension checkpoint

- Which of the following calculations are correct:

$$\lim_{x\to 1}\frac{1}{x-1} = \lim_{x\to 1}\frac{\frac{d(1)}{dx}}{\frac{d}{dx}(x-1)} = \lim_{x\to 1}\frac{0}{1} = 0$$

$$\lim_{x\to 0}\frac{\sin 2x}{\tan 3x} = \lim_{x\to 0}\frac{\frac{d}{dx}\sin 2x}{\frac{d}{dx}\tan 3x} = \lim_{x\to 0}\frac{2\cos 2x}{\frac{3}{\cos^2 3x}} = \frac{2}{3}$$

$$\lim_{x\to 0}(\cos x)^{1/x} = \lim_{x\to 0} e^{\frac{1}{x}\ln(\cos x)} = e^{\lim\limits_{x\to 0}\frac{\ln(\cos x)}{x}} = e^{\lim\limits_{x\to 0}\frac{-\sin x}{\cos x}} = e^0 = 1$$