Lecture 21

Implicit Differentiation

Objectives

In this lecture we will learn how an equation in two variables may define a function in one variable.

We will learn how to **differentiate** such a function,
even if we do not know a formula which defines this function explicitly.

We will learn how to find the equation of a **tangent line** to a curve
which is not the graph of a function.

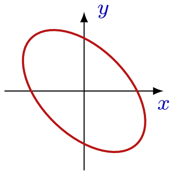
Implicit vs Explicit

Explicit means expressed clearly, without any ambiguity.
For example: *The professor gave explicit instructions for the midterm.*

Implicit means not directly expressed, but to be understood.
For example: *His speech contained an implicit criticism of the government.*

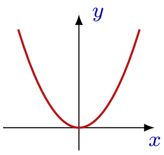
So far, we studied functions $y = f(x)$ given by an **explicit** formula for f, like $f(x) = x^2 + x$.
However, there are many situations in which the explicit formula is not known or even does not exist and a function is defined in a more complicated way.

It may happen that the independent variable x and the dependent variable y are related by an equation $F(x, y) = 0$ in which x and y are involved equally, like in $x^2 + y^2 + xy - 1 = 0$.
Geometrically, the equation $F(x, y) = 0$ represents a **curve** on the xy-plane.
For example, $x^2 + y^2 + xy - 1 = 0$ represents an **ellipse**.
This curve is **not** the graph of a function,
since it fails the vertical line test.
There is no convenient explicit formula for y as a function of x.
But for many purposes, the equation $F(x, y) = 0$ is most convenient.
We just need to learn how to use it.



Equations and functions

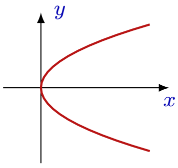
Example 1. Consider the **equation** $x^2 - y = 0$. It defines a parabola:



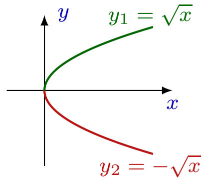
The parabola is the graph of a function $y(x) = x^2$.
We say that the equation $x^2 - y = 0$ defines
the function $y(x) = x^2$ **implicitly**,
while the equation $y = x^2$ defines it **explicitly**.

The graph of the equation $x^2 - y = 0$ is indeed the graph of a function
since it satisfies the conditions of the vertical line test.

Example 2. Consider the equation $x - y^2 = 0$. It also defines a parabola:



The graph of the equation is **not** a graph of any function,
since it fails the vertical line test.
So the equation $x - y^2 = 0$ does **not** define a single function $y = y(x)$.



Nevertheless, $x - y^2 = 0$ defines two different functions:
$y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$.

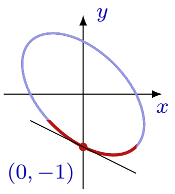
Functions defined implicitly

It may be difficult or even impossible to find an **explicit** formula
for a function $y = y(x)$ defined by the **implicit** equation $F(x, y) = 0$:

$$F(x, y) = 0 \overset{?}{\Longrightarrow} y = y(x) \quad ☹$$

However, many properties of a function $y = y(x)$ defined **implicitly** by the equation $F(x, y) = 0$ can be found without the explicit form for $y = y(x)$.

Example. The equation $x^2 + y^2 + xy - 1 = 0$ can't be solved for y in terms of x: there is no function such that the ellipse $x^2 + y^2 + xy - 1 = 0$ is the graph of this function.



Choose a point on the ellipse, say $(0, -1)$.
In a neighborhood of this point,
the equation defines **implicitly** a function $y = y(x)$:
$x^2 + y^2 + xy - 1 = 0 \implies y = y(x)$.
Its graph lies on the ellipse.

We will find the derivative $\dfrac{dy}{dx}$ directly from the equation of ellipse **without** writing down $y(x)$ explicitly.
The derivative will represent the slope of the tangent to the ellipse.

The slope by implicit differentiation

Problem. Find the slope of the tangent line to the ellipse $x^2 + y^2 + xy = 1$ at the point $(0, -1)$.

Solution. The slope of the tangent line at $(0,-1)$ is $\left.\dfrac{dy}{dx}\right|_{\substack{x=0\\y=-1}}$, where $y = y(x)$ is the function defined by the equation $x^2 + y^2 + xy = 1$ **implicitly**.

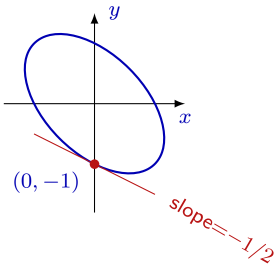
We find this derivative by **implicit differentiation**.
Let us rewrite the equation replacing y by $y(x)$:
$x^2 + y^2(x) + x \cdot y(x) = 1$. Differentiate this equation with respect to x.
Keep in mind that $y^2(x)$ should be differentiated by the chain rule as a **composition** of two functions:

$x^2 + y^2(x) + x \cdot y(x) = 1 \xrightarrow{\frac{d}{dx}} 2x + 2y(x)\cdot\dfrac{dy}{dx} + y(x) + x \cdot \dfrac{dy}{dx} = 0$. Or, equivalently,
$2x + 2yy' + y + xy' = 0$. When $x = 0,\ y = -1$, we get
$2 \cdot 0 + 2(-1)y' + (-1) + 0 \cdot y' = 0 \implies y' = \boxed{-1/2}$.

The slope by implicit differentiation

We have found the derivative of a function without knowing an **explicit** formula for the function!

The slope of the tangent line to the ellipse $x^2 + y^2 + xy = 1$ at the point $(0,-1)$ is $\left.\dfrac{dy}{dx}\right|_{\substack{x=0\\y=-1}} = -\dfrac{1}{2}$.



Implicit differentiation

Problem. Find $\dfrac{dy}{dx}$ if $x - y^2 = 0$.

Solution. We know that the implicit equation $x - y^2 = 0$ defines two functions, $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$. Their derivatives are $\dfrac{dy_1}{dx} = \dfrac{1}{2\sqrt{x}}$ and $\dfrac{dy_2}{dx} = -\dfrac{1}{2\sqrt{x}}$.

However, we may find $\dfrac{dy}{dx}$ without solving the equation $x - y^2 = 0$ for y.
Let us rewrite the equation replacing y by $y(x)$:
$x - y^2(x) = 0$. Differentiate this equation. Keep in mind that $y^2(x)$ should be differentiated by the chain rule as a **composition** of two functions.

$$x - y^2(x) = 0 \xrightarrow{\frac{d}{dx}} 1 - 2y\,\frac{dy}{dx} = 0 \implies \boxed{\frac{dy}{dx} = \frac{1}{2y}}$$

Notice that the derivative is given in terms of y

This formula agrees with the derivatives calculated for both of the explicit solutions $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$:

$$\frac{dy_1}{dx} = \frac{1}{2y_1} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad \frac{dy_2}{dx} = \frac{1}{2y_2} = -\frac{1}{2\sqrt{x}}.$$

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Tangent line by implicit differentiation

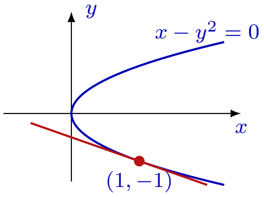
Problem. Find the equations of the tangent line to the curve $x - y^2 = 0$ at the point $(1, -1)$.

Solution. The equation of the tangent line to the curve $y = y(x)$ at the point (x_0, y_0) is $y - y_0 = y'(x_0)(x - x_0)$.
In our case, $x_0 = 1$ and $y_0 = -1$. What is $y'(x_0)$ then?

By implicit differentiation, $x - y^2(x) = 0 \implies 1 - 2y\,\dfrac{dy}{dx} = 0 \implies \dfrac{dy}{dx} = \dfrac{1}{2y}$

Therefore, $y'(x_0) = \left.\dfrac{dy}{dx}\right|_{\substack{x = x_0 \\ y = y_0}} = \left.\dfrac{1}{2y}\right|_{\substack{x = 1 \\ y = -1}} = -\dfrac{1}{2}$.

Hence the equation of the tangent line is $y - (-1) = -\dfrac{1}{2}(x - 1) \iff \boxed{y = -\dfrac{1}{2}x - \dfrac{1}{2}}$



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Tangent to a circle

Problem. Find the slope of the tangent line to the circle $x^2 + y^2 = 25$ at the point $(-3, 4)$.

Solution. The slope of the tangent at $(-3, 4)$ is $\left.\dfrac{dy}{dx}\right|_{\substack{x=-3\\ y=4}}$.

The derivative can be found by the **implicit differentiation**.

Let $y = y(x)$ be a function defined by the equation $x^2 + y^2 = 25$.

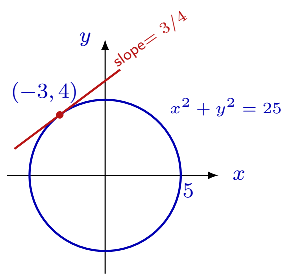
Then $x^2 + y^2(x) = 25$.

Differentiate this equation implicitly with respect to x:

$2x + 2y\dfrac{dy}{dx} = 0 \implies \dfrac{dy}{dx} = -\dfrac{x}{y}$. So $\left.\dfrac{dy}{dx}\right|_{\substack{x=-3\\ y=4}} = -\left.\dfrac{x}{y}\right|_{\substack{x=-3\\ y=4}} = -\dfrac{-3}{4} = \dfrac{3}{4}$.

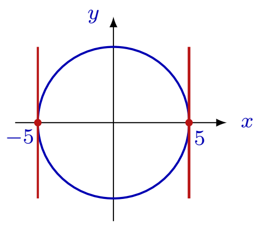
Therefore, the slope is $\dfrac{3}{4}$.

Tangent to a circle



The slope of the tangent line to the circle $x^2 + y^2 = 25$ at $(-3, 4)$ is $3/4$.

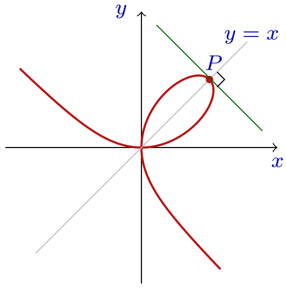
Control question.
For which points on the circle is the tangent **vertical**?
As we see from the picture, for $(-5, 0)$ and $(5, 0)$.
These are the only points where $\dfrac{dy}{dx}$ is undefined.



Indeed, $\dfrac{dy}{dx} = -\dfrac{x}{y}$ and is undefined if $y = 0$.

Folium of Descartes

The equation $x^3 + y^3 - 6xy = 0$ defines a curve on the xy-plane.
It is called the *folium of Descartes*:



The folium of Descartes is symmetric about the line $y = x$.
Let us show that the tangent line to the folium at the point P is orthogonal (perpendicular) to the line $y = x$.

Folium of Descartes

Solution. We have to show that the slope of the tangent line at P and the slope of $y = x$ are negative reciprocals of each other.

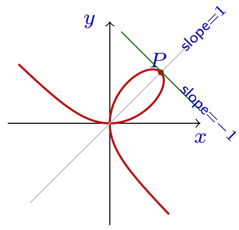
The slope of the tangent can be found by the **implicit differentiation** of the equation of the folium:

$x^3 + y^3(x) - 6xy(x) = 0 \xrightarrow{\frac{d}{dx}} 3x^2 + 3y^2(x)\dfrac{dy}{dx} - 6y(x) - 6x\dfrac{dy}{dx} = 0$, that is

$x^2 + y^2y' - 2y - 2xy' = 0$. From which we get $y' = \dfrac{2y - x^2}{y^2 - 2x}$.

The point P belongs the line $y = x$, so $y = x$ at P.

Therefore, $y'\Big|_P = \dfrac{2x - x^2}{x^2 - 2x} = -1$,
and the slope of the tangent is -1.
Therefore, the tangent line is perpendicular to $y = x$.



Second derivative by implicit differentiation

Example. Find $y'' = \dfrac{d^2y}{dx^2}$ if $x^2 + y^2 = 1$.

Solution. Differentiate implicitly the equation $x^2 + y^2 = 1$ and get
$2x + 2yy' = 0$, that is $x + yy' = 0$. $(*)$

Differentiate the obtained equation one more time:

$1 + y'y' + yy'' = 0$, or, equivalently, $1 + (y')^2 + yy'' = 0$.

Solve for y'': $y'' = -\dfrac{1 + (y')^2}{y}$. From $(*)$ we get $y' = -\dfrac{x}{y}$.

Therefore, $y'' = -\dfrac{1 + (y')^2}{y} = -\dfrac{1 + \left(-\dfrac{x}{y}\right)^2}{y} = -\dfrac{x^2 + y^2}{y^3} = \boxed{-\dfrac{1}{y^3}}$, since $x^2 + y^2 = 1$.

Riddle. How to interpret the obtained result geometrically?

Finding maximum and minimum on the curves

Problem. Find local extrema on the curve $-x^2 + y^2 = 1.$
Solution. We search for local extema among **critical** and **singular** points of a function $y = y(x)$ defined implicitly by the equation $-x^2 + y^2 = 1$.

Implicit differentiation gives us

$-x^2 + y^2 = 1 \xrightarrow{\frac{d}{dx}} -2x + 2y\dfrac{dy}{dx} = 0 \implies \dfrac{dy}{dx} = \dfrac{x}{y}$.

Critical points: $\dfrac{dy}{dx} = \dfrac{x}{y} = 0 \iff x = 0$. In this case, $-0^2 + y^2 = 1 \implies y = 1$ or $y = -1$.

Therefore, the critical points are $(0, 1)$ and $(0, -1)$.

Singular points: $\dfrac{dy}{dx}$ doesn't exist $\iff y = 0$. In this case, $-x^2 + 0^2 = 1$.

There are no such x on the curve, therefore there are no singular points.
To classify the critical points $(0, 1)$ and $(0, -1)$, we apply the **second derivative test**.

Second derivative test

After the first implicit differentiation, we have

$$-x^2 + y^2 = 1 \xrightarrow{\frac{d}{dx}} -2x + 2y\frac{dy}{dx} = 0 \iff -x + yy' = 0.$$

Differentiate the latter equation once more:

$$-x + yy' = 0 \xrightarrow{\frac{d}{dx}} -1 + y' \cdot y' + y \cdot y'' = 0 \iff -1 + (y')^2 + yy'' = 0.$$

At a critical point, $y' = 0$. Therefore, $-1 + yy'' = 0 \qquad (*)$
Calculate y'' at the critical points $(0, 1)$ and $(0, -1)$.

Plug $x = 0$ and $y = 1$ in $(*)$: $-1 + 1 \cdot y'' = 0 \implies y''\Big|_{(0,1)} = 1 > 0$.

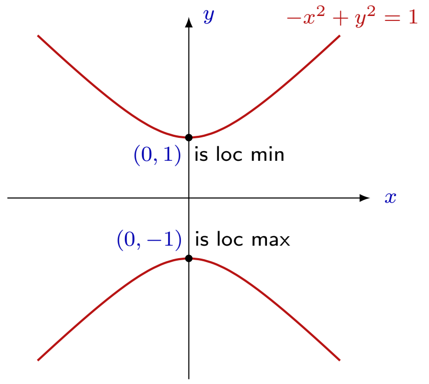
Therefore, $(0, 1)$ is a local minimum.

Plug in $x = 0$ and $y = -1$ in $(*)$: $-1 - 1 \cdot y'' = 0 \implies y''\Big|_{(0,1)} = -1 < 0$.

Therefore, $(0, -1)$ is a local maximum.

Hyperbola

The curve $-x^2 + y^2 = 1$ is a hyperbola:



How to get dx/dy ?

Problem. Show that the point $(1,0)$ belongs to the curve $1+\sin(xy)=x+y$ and find $\dfrac{dx}{dy}$ at $(1,0)$.

Solution. For $x=1$ and $y=0$ the equation turns into
a true numerical identity: $1+\sin(1\cdot 0)=1+0 \iff 1=1$ ✓
Therefore, the point $(1,0)$ belongs to the curve.

The equation $1+\sin(xy)=x+y$ defines **implicitly** a function $x=x(y)$. We have to find the derivative of this function when $x=1$ and $y=0$.

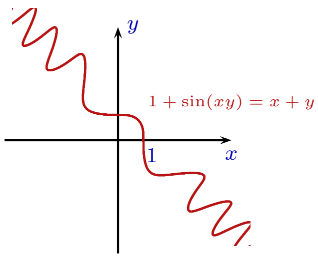
Differentiate $1+\sin(xy)=x+y$ implicitly with respect to y:

$$1+\sin(x(y)\cdot y)=x(y)+y \xrightarrow{\frac{d}{dy}} \cos(x(y)\cdot y)\left(\frac{dx}{dy}y+x(y)\cdot 1\right)=\frac{dx}{dy}+1\,.$$

In other words, $(x'y+x)\cos(xy)=x'+1$. Plug in $x=1$ and $y=0$:

$(x'\cdot 0+1)\cos(1\cdot 0)=x'+1 \implies x'=0$. So $\left.\dfrac{dx}{dy}\right|_{\substack{x=1\\ y=0}}=0$.

What is dx/dy geometrically?



We have got that $\dfrac{dx}{dy}=0$ at $(1,0)$.

Since $\dfrac{dy}{dx}=\dfrac{1}{dx/dy}$, then $\dfrac{dy}{dx}=\infty$ at $(1,0)$,

and the curve has the vertical tangent line at $(1,0)$.

Summary/Comprehension checkpoint

In this lecture we learned how to differentiate a function defined **implicitly**.

• Show that the point $(1, 0)$ belongs to the curve $x\ln(x^2 + y^2) + y = 0$ and find the equation of the tangent line to the curve at this point.

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