Lecture 20

The Second Derivative Test

Objectives

In this lecture we will learn how to use the **second** derivative of a function to find intervals of **concavity** and **inflection** points.

We will apply the second derivative test to the **classification** of local extrema.

We will see how to use information about the second derivative to draw the **graph** of a function.

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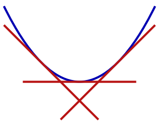
Concavity

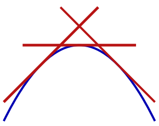
Definition. A differentiable function f is *concave up* on an interval if f' **increases** on the interval.



In this case, the graph is located **above** the tangents at all points on the interval.

If f'' exists, then f is concave up if and only if $f'' > 0$.

A function f is *concave down* on an interval if f' **decreases** on the interval.



In this case, the graph is located **below** the tangents at all points on the interval.

If f'' exists, then f is concave down if and only if $f'' < 0$.

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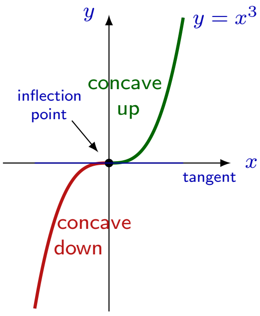
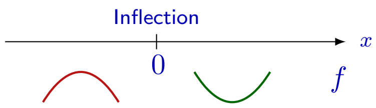
Inflection points

Definition. An *inflection point* is a point where the graph **changes** concavity.

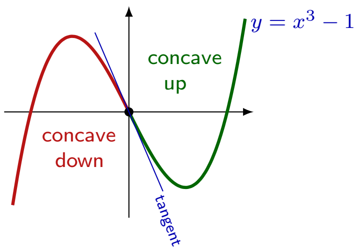
Example. $x = 0$ is an **inflection point** for the function $f(x) = x^3$.

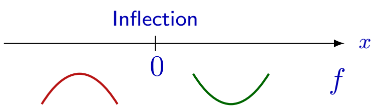


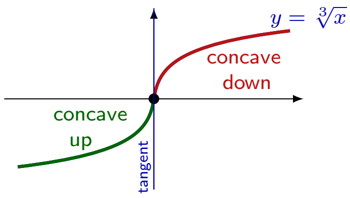


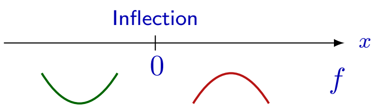
☞ At an inflection point,
the graph passes from one side of the tangent line to the other.

Examples of inflection points







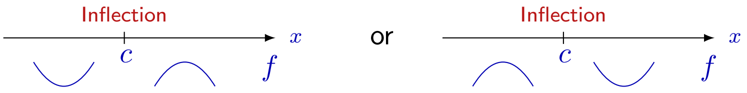


Remark. $y = \sqrt[3]{x}$ is not differentiable at $x = 0$.

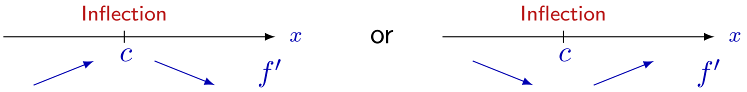
The second derivative at an inflection point

Theorem. If c is an inflection point and $f''(c)$ exists, then $f''(c) = 0$.

Proof. If c is an inflection point, then f is concave up to the left of c and concave down to the right from c, or vice versa:



Therefore, f' increases to the left from c and decreases to the right from c (or vice versa):



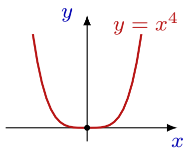
It follows that f' has a local maximum or local minimum at c, and, therefore, by Fermat' theorem, $f''(c) = 0$. $\square$

What if $f''(c) = 0$, and what if $f''(c)$ does not exist?

We have proven that if c is an inflection point then $f''(c) = 0$.
The converse is **not** true.

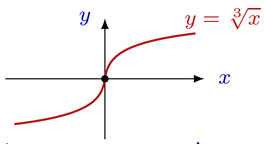
For example, $f(x) = x^4$ does not have an inflection point at $x = 0$ though

$$f''(0) = \frac{d^2(x^4)}{dx^2}\bigg|_{x=0} = 12x^2\bigg|_{x=0} = 0\,.$$



It may happen that c is an inflection point, but $f''(c)$ does not exist.

For example, $f(x) = \sqrt[3]{x}$ has an inflection point at $x = 0$



but $f''(x) = \dfrac{d^2(x^{1/3})}{dx^2} = \left(\dfrac{1}{3}x^{-2/3}\right)' = -\dfrac{2}{9}x^{-5/3} = -\dfrac{2}{9\sqrt[3]{x^5}}$ does not exist at 0.

Using concavity and inflection points for graphing

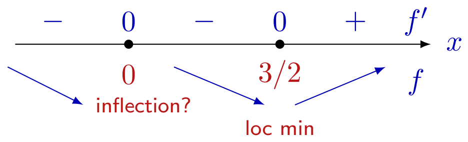
Example 1. For the function $f(x) = x^4 - 2x^3 + 1$, find
- local extrema and determine their types (local maximum or local minimum),
- intervals of increase and interval of decrease,
- intervals of up and down concavity,
- inflection points.

Using this information, draw the graph of the function.

Solution. Calculate the **first** derivative: $f'(x) = 4x^3 - 6x^2 = 2x^2(2x-3)$.
Find the **critical points** (where $f'(x) = 0$):

$f'(x) = 0 \iff 2x^2(2x-3) = 0 \iff x = 0$ or $x = 3/2$.

Do the **first derivative test**:

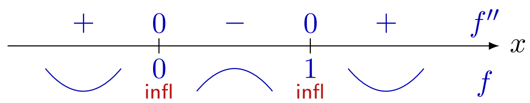


Information from the second derivative

Calculate the **second derivative**:

$f''(x) = (4x^3 - 6x^2)' = 12x^2 - 12x = 12x(x-1)$.

Determine the **sign** of f'', the intervals of **concavity** and the **inflection points**:



For graphing, we need the **values** of f at the local minimum $x = 3/2$
and at the inflection points $x = 0$ and $x = 1$:

$f(3/2) = (3/2)^4 - 2(3/2)^3 + 1 = -11/16$,

$f(0) = 0^4 - 2 \cdot 0^3 + 1 = 1,$

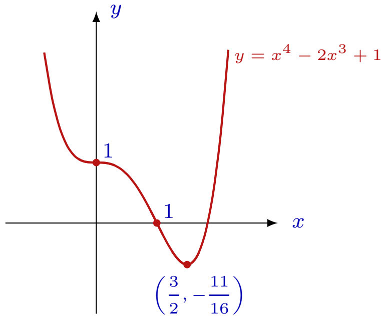
$f(1) = 1^4 - 2 \cdot 1^3 + 1 = 0.$

Putting everything together

What do we know about our function $f(x) = x^4 - 2x^3 + 1$?

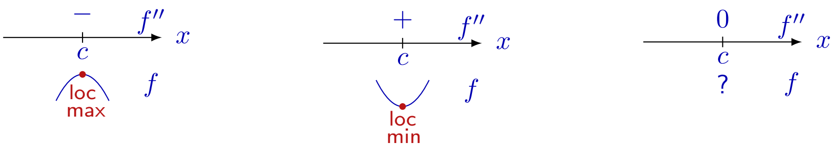
- f **decreases** on $(-\infty, 1]$ and **increases** on $[1, \infty]$
- $(3/2, -11/16)$ is a **local minimum**
- $(0,1)$ and $(1,0)$ are **inflection points**
- f is **concave up** on $(-\infty, 0]$ and $[1, \infty)$, and **concave down** on $[0,1]$.

Using this information, we can draw the graph:



The second derivative test

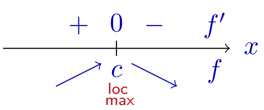
Theorem. Let c be a **critical point** of f, that is, $f'(c) = 0$.
If $f''(c) < 0$ then c is a **local maximum**,
if $f''(c) > 0$ then c is a **local minimum**,
if $f''(c) = 0$ then the test is inconclusive.



Proof. Let $f'(c) = 0$ and $f''(c) < 0$. Then

$$\lim_{h \to 0} \frac{f'(c+h)}{h} = \lim_{h \to 0} \frac{f'(c+h) - f'(c)}{h} = f''(c) < 0.$$

Therefore, $f'(c+h) < 0$ for all sufficiently small positive h
and $f'(c+h) > 0$ for all sufficiently small negative h.



By the first derivative test, c is a local maximum.
The case of $f''(c) > 0$ is handled similarly. $\square$

The second derivative test: example

Example. Find and classify the critical points of $f(x) = x^2 e^{-x}$.

Solution. $f'(x) = 2xe^{-x} - x^2 e^{-x} = (2x - x^2)e^{-x} = x(2 - x)e^{-x}$.

The **critical points** are $x = 0$ and $x = 2$.

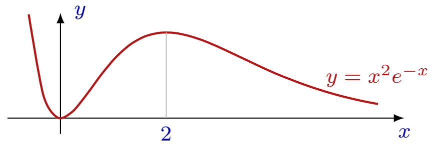
To apply the **second derivative test**, we calculate

$f''(x) = (2 - 2x)e^{-x} - (2x - x^2)e^{-x} = (2 - 4x + x^2)e^{-x}$.

Determine the **signs** of $f''(0)$ and $f''(2)$:

$f''(0) = 2e^{-0} = 2 > 0$, $f''(2) = -2e^{-2} < 0$.

Therefore $x = 0$ is a **local minimum** and $x = 2$ is a **local maximum**.



First derivative test vs. Second derivative test

Both the first derivative test and the second derivative test are used to classify the critical points of a function.

Which one is better?

When we need to perform a **complete study** of a function (including finding intervals of increase/decrease and graphing), the first derivative test is preferable.

Also, the first derivative test allows us to classify **singular points**, for which the second derivative test does not work.

If the task is restricted to the classification of critical points and the second derivative exists (and is not too difficult to calculate), then the second derivative test may be more efficient.

Drawing functions with singular points

Example. For the function $f(x) = x^{2/3}(6-x)^{1/3}$, find
- local extrema and determine their types,
- intervals of increase and interval of decrease,
- intervals of concavity,
- inflection points.

Using this information, draw the graph of the function.

Solution. Use **logarithmic differentiation** to find f':

$$y = x^{2/3}(6-x)^{1/3} \implies \ln y = \frac{2}{3}\ln x + \frac{1}{3}(6-x).$$

$$\frac{y'}{y} = \frac{2}{3x} - \frac{1}{3(6-x)} = \frac{2(6-x)-x}{3x(6-x)} = \frac{12-3x}{3x(6-x)} = \frac{4-x}{x(6-x)}.$$

Therefore, $y' = y\dfrac{4-x}{x(6-x)} = \dfrac{4-x}{x^{1/3}(6-x)^{2/3}}$.

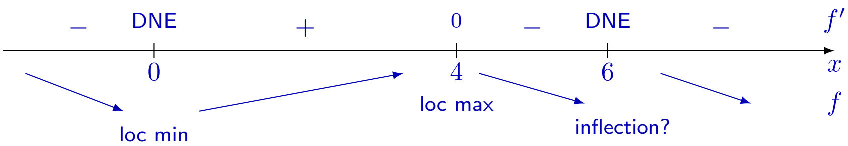
Working on the example: first derivative test

We have found $f'(x) = \dfrac{4-x}{x^{1/3}(6-x)^{2/3}}$.

The **critical point** of f (where $f'(x) = 0$) is $x = 4$.

The **singular points** of f (where $f'(x)$ is undefined) are $x = 0$ and $x = 6$.

The **first derivative test:**



f **decreases** on $(-\infty, 0)$, $[4, 6)$ and $(6, \infty)$, and **increases** on $(0, 4]$.

$x = 4$ is a **local maximum**, $f(4) = 4^{2/3}(6-4)^{1/3} = 2^{5/3}$.

The graph has **vertical** tangent lines at the singular points $x = 0$ and $x = 6$.

$x = 0$ and $x = 6$ are the x-intercepts of the graph since $f(0) = 0$, $f(6) = 0$.

Working on the example: second derivative

To investigate concavity, we have to calculate the second derivative.

Since $f'(x) = \dfrac{4-x}{x^{1/3}(6-x)^{2/3}}$,

we calculate $f''(x)$ using logarithmic differentiation.

Let $u = \dfrac{4-x}{x^{1/3}(6-x)^{2/3}} = (4-x)x^{-1/3}(6-x)^{-2/3}$, then $\ln u = \ln(4-x) - \dfrac{1}{3}\ln x - \dfrac{2}{3}\ln(6-x)$, and

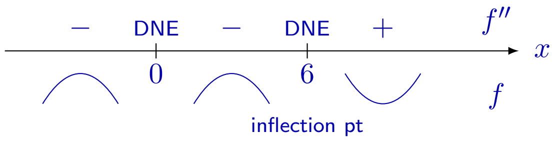
$\dfrac{u'}{u} = -\dfrac{1}{4-x} - \dfrac{1}{3x} + \dfrac{2}{3(6-x)} = \cdots = \dfrac{-8}{x(4-x)(6-x)}$. So

$$u' = u\left(-\frac{8}{x(4-x)(6-x)}\right) = \frac{4-x}{3x^{1/3}(6-x)^{2/3}}\left(\frac{-8}{x(4-x)(6-x)}\right)$$

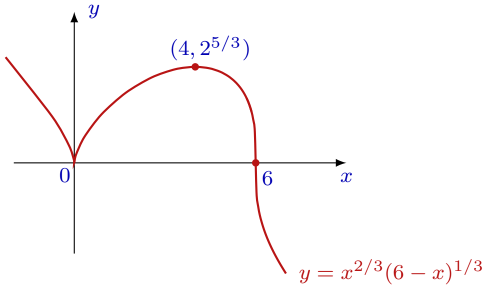
$= \dfrac{-8}{x^{4/3}(6-x)^{5/3}}$. By this, since $u = f'(x)$, this means $f''(x) = \dfrac{-8}{x^{4/3}(6-x)^{5/3}}$.

Working on the example: concavity

The sign of $f''(x) = \dfrac{-8}{x^{4/3}(6-x)^{5/3}}$ determines concavity:



The graph is **concave down** on $(-\infty, 0)$ and $(0, 6)$, and **concave up** on $(6, \infty)$.



Summary

In this lecture we learned

- what it means for a function to be **concave up** or **concave down** on an interval
- what an **inflection** point is
- how to use the **second derivative test** to determine the types of local extrema
- how to use the first and second derivative tests to analyze functions.

Comprehension checkpoint

- Let $x = 1$ be an inflection point of $f(x)$. Is it true that $f''(1) = 0$?
- Let $f''(3) = 0$. Is it true that $x = 3$ is an inflection point of f?
- Let $f''(x) > 0$ on $[1, 2]$. Is it true that f increases on $[1, 2]$?
- Let $f''(x) > 0$ on $[1, 2]$. Is it true that f' increases on $[1, 2]$?
- Let $f''(x) > 0$ on $[1, 2]$. Is it true that f is concave up $[1, 2]$?
- Let $f''(1) = 0$, $f''(x) > 0$ on $[0, 1)$ and $f''(x) < 0$ on $(1, 2]$.
Is it true that $x = 1$ is an inflection point of f?