Lecture 19

First Derivative Test

Objectives

In this lecture we will learn how to use **the first derivative** of a function to explore the function's behavior, namely how

- to find intervals where the function increases and intervals where it decreases
- to find and classify critical and singular points.

We will see how to use the information to draw the **graph** of a function.

Increasing and decreasing functions

Definition. Let f be a function defined on an interval I.

f is called *increasing* on I if $f(x_1) < f(x_2)$ whenever $x_1, x_2 \in I$ and $x_1 < x_2$.

f is called *decreasing* on I if $f(x_1) > f(x_2)$ whenever $x_1, x_2 \in I$ and $x_1 < x_2$.

Example. $f(x) = x^2$ decreases on $(-\infty, 0]$ and increases on $[0, \infty)$.

Indeed, for any $x_1 < x_2 \leq 0$, we have

$$f(x_1) - f(x_2) = x_1^2 - x_2^2 = \underbrace{(x_1 - x_2)}_{<0}\underbrace{(x_1 + x_2)}_{<0} > 0\,.$$

So $f(x_1) > f(x_2)$. Thus f **decreases** on $(-\infty, 0]$. For any $0 \leq x_1 < x_2$, we have

$$f(x_1) - f(x_2) = x_1^2 - x_2^2 = \underbrace{(x_1 - x_2)}_{<0}\underbrace{(x_1 + x_2)}_{>0} < 0\,.$$

So $f(x_1) < f(x_2)$. Thus f **increases** on $[0, \infty)$.

The Increasing/decreasing test

To check from the definition whether a function is increasing/decreasing may be cumbersome. The theorem below gives a simple and convenient criterion for increasing/decreasing in terms of the derivative.

Theorem. If $f'(x) > 0$ for all x in an interval, then f is **increasing** on that interval.
If $f'(x) < 0$ for all x in an interval, then f is **decreasing** on that interval.

Proof. Assume that $f'(x) > 0$ for all x in an interval.

Choose any two points x_1, x_2 in the interval with $x_1 < x_2$.

Then, by the Mean Value Theorem, there exists a point x in the interval such that

$f(x_2) - f(x_1) = f'(x)(x_2 - x_1)$.

Since $f'(x) > 0$ and $x_2 - x_1 > 0$, this means that $f(x_2) - f(x_1) > 0$,

that is $f(x_2) > f(x_1)$. Therefore, f is increasing.

For $f'(x) < 0$, the proof is similar. □

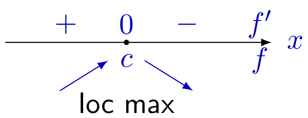
The first derivative test at a critical point

Definition. A point x is called a *critical point* of f if $f'(x) = 0$.
The following theorem follows directly from the increasing/decreasing test.

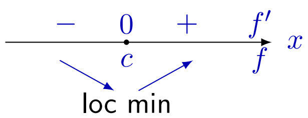
Theorem (First derivative test).

Let c be a **critical point** of a differentiable function f, that is, $f'(c) = 0$.

- If the **sign** of f' changes from $+$ to $-$ at c, then c is a **local maximum**:



- If the **sign** of f' changes from $-$ to $+$ at c, then c is a **local minimum**:



- If the sign of f' does not change at c, then c is neither local maximum nor local minimum.

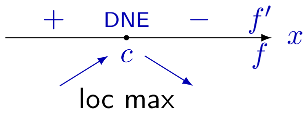
The first derivative test at a singular point

Definition. A point x in the domain of a function f is called a *singular point* of f if $f'(x)$ does not exist (that is $f(x)$ is defined, but $f'(x)$ is not.)

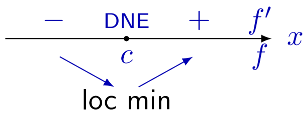
Theorem (First derivative test).

Let c be a **singular point** of a continuous function f (i.e. $f'(c)$ does not exist) and assume that f is differentiable at all x near c excluding c.

- If the **sign** of f' changes from $+$ to $-$ at c, then c is a **local maximum**:



- If the **sign** of f' changes from $-$ to $+$ at c, then c is a **local minimum**:



- If the sign of f' doesn't change at c, then c is nether local maximum nor local minimum.

First derivative test: example

Problem. For the function $f(x) = x^3 - 3x$, find
- local extrema and determine their types (local maximum or local minimum),
- the intervals of increase and intervals of decrease,
- draw the graph of the function.

Solution. We know that if a function is differentiable,
then the derivative vanishes at local extreme points (Fermat's theorem).

The function is differentiable, so our plan of our solution is the following:

1. Find the **critical points** (CP) of f, that is the points where $f'(x) = 0$.
The extreme points are among them.

2. Apply the **first derivative test** to determine
the intervals of increase/decrease and the types of the extreme points.

3. Use information from **1** and **2** to draw the graph of the function.

Example

We start with finding the **critical points** of $f(x) = x^3 - 3x$:

$$f'(x) = 0 \iff 3x^2 - 3 = 0 \iff x^2 - 1 = 0 \iff (x-1)(x+1) = 0$$
$$\iff x = -1 \text{ or } x = 1 .$$

So there are two critical points: $x = -1$ and $x = 1$. To determine whether they are extreme points and find the intervals of increase/decrease,
we perform the **first derivative test**.

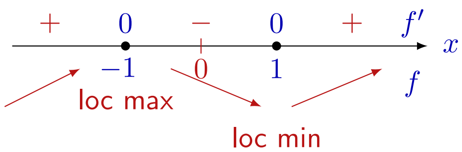
To determine the **sign** of f', we have to solve the inequalities $f' > 0$ and $f' < 0$. We solve the inequalities by the **method of intervals**.

On the real axis, we place the critical points:

Two critical points split the real axis into three intervals.
The sign of the derivative stays unchanged on each interval.

The method of intervals

To determine the sign of derivative on each interval,
we choose a test point x within the interval and determine the sign of $f'(x)$.



For example, $f'(0) = 3x^2 - 3\Big|_{x=0} = -3 < 0$.

In a similar way we determine the sign of f' on the other two intervals.

Using the **increasing/decreasing test**, we find the intervals of increase/decrease.

f increases on $(-\infty, -1]$ and $[1, \infty)$, decreases on $[-1, 1]$.
Now apply now the **first derivative test** to determine the types of the critical points.

Then we will be ready to draw the graph of the function.

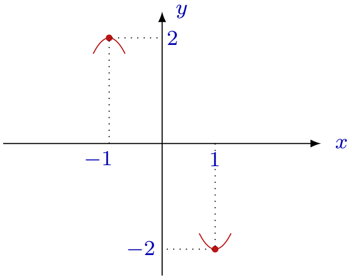
Drawing the graph: beginning

First of all, we determine the locations of the extreme points on the graph.

The value of the function $f(x) = x^3 - 3x$ at the **local maximum** $x = -1$ is $f(-1) = (-1)^3 - 3(-1) = -1 + 3 = 2$.

The value of f at the **local minimum** $x = 1$ is $f(1) = 1^3 - 3 \cdot 1 = -2$. Therefore,

f has a $\boxed{\text{local maximum at } (-1, 2)}$ and a $\boxed{\text{local minimum at } (1, -2)}$.



Draw a small "hat" at $(-1, 2)$ and a small "cup" at $(1, -2)$.

Drawing the graph: completion

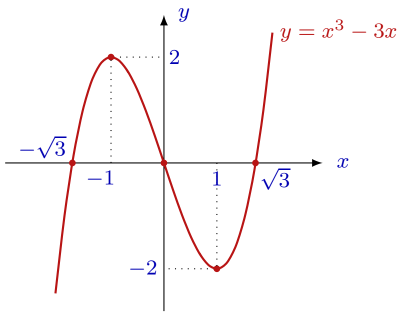
To continue the drawing, we observe that $f(x) = x^3 - 3x$ is an **odd** function since $f(-x) = -f(x)$ for any x:

$f(-x) = (-x)^3 - 3(-x) = -x^3 + 3x = -(x^3 - 3x) = -f(x)$.

Therefore, the graph of f is **symmetric** about the origin.

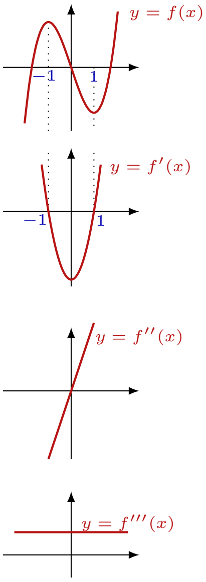
We locate the x-**intercepts** of the graph by solving $f(x) = 0$:

$f(x) = 0 \iff x^3 - 3x = 0 \iff x(x^2 - 3) = 0 \iff x = 0$ or $x = \sqrt{3}$ or $x = -\sqrt{3}$.



Using this information, we can complete our sketch of the graph.

Graphing a function and its derivatives



Compare the graphs of the function $f(x) = x^3 - 3x$ and of its first three derivatives

$f'(x) = 3x^2 - 3\,,\ f''(x) = 6x\,,\ f'''(x) = 6:$

$f \nearrow \iff f' > 0 \iff x \in (-\infty, -1]$ or $x \in [1, \infty)$
$f \searrow \iff f' < 0 \iff x \in [-1, 1]$

$f' \nearrow \iff f'' > 0 \iff x \in [0, \infty)$
$f' \searrow \iff f'' < 0 \iff x \in (-\infty, 0]$

$f'' \nearrow \iff f''' > 0 \iff x \in (-\infty, \infty)$

Don't mix up a function and its derivative!

Graphing functions with singular points

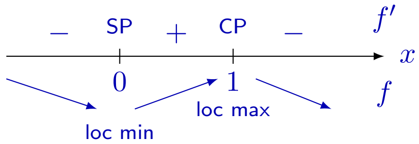
Example. For the function $f(x) = 3x^{2/3} - 2x$, find
- local extrema and determine their types (local maximum or local minimum),
- the intervals of increase and intervals of decrease,
- draw the graph of the function.

Solution. $f'(x) = (3x^{2/3} - 2x)' = 2x^{-1/3} - 2 = 2\left(\dfrac{1 - \sqrt[3]{x}}{\sqrt[3]{x}}\right).$

Critical points: $f'(x) = 0 \iff x = 1$.

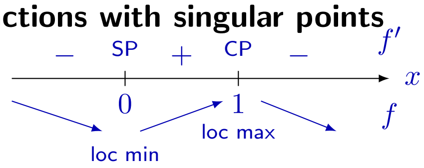
Singular points: $f'(x)$ does not exist $\iff x = 0$.

First derivative test:



At $x = 0$, f has a vertical tangent line, since $\lim\limits_{x \to 0} |f'(x)| = \infty$.

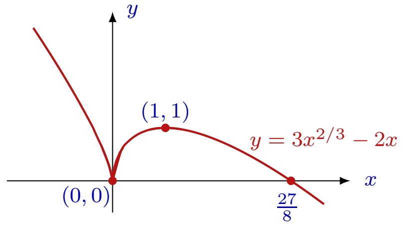
Graphing functions with singular points



From the first derivative test, we are obtain the following information:

f increases on $[0,1]$, f decreases on $(-\infty,0]$ and $[1,\infty)$.
f has local minimum at $x=0$, $f(0)=3\cdot 0^{2/3}-2\cdot 0=0$,
⚠ **Warning:** at $(0,0)$, the graph has a vertical tangent line.
f has a local maximum at $x=1$, $f(1)=3\cdot 1^{2/3}-2\cdot 1=1$.

x-intercepts: $f(x)=0 \iff 3x^{2/3}-2x=0 \iff$
$2x^{2/3}\left(3/2-x^{1/3}\right)=0 \iff x=0$ or $x=27/8$.
With this information, we can make a sketch of the graph.



14 / 16

Summary

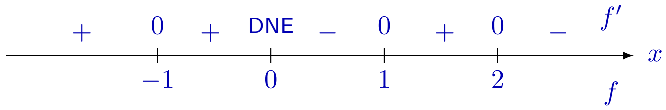
In this lecture we learned how to apply the **first derivative test** to a function

- to find and classify the critical and singular points of the function
- to find the function's intervals of increase and intervals of decrease.

15 / 16

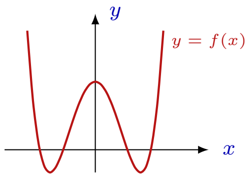
Comprehension checkpoint

- From the first derivative test, you obtained the following information about a function $y = f(x)$:



(The values written under the axis are the values of x, not f.)
Additionally, you know that $f(-1) = f(1) = 1,\quad f(0) = 3$ and $f(2) = 3.$
Sketch the graph of $y = f(x)$.

- Given this graph for $y = f(x)$,
draw the graphs of $y = f'(x),\ \ y = f''(x),\ \ y = f'''(x)$ and $y = f^{(4)}(x)$:



16 / 16